CHERRY FIELDS AND THE ROTATION NUMBERS OF ONE PARAMETER FAMILIES

OF MAPS OF THE CIRCLE

by

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Contents

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It hardly needs stating how much I owe to my parents.

The Science and Engineering Research Council has been good enough to support me financially during my efforts in the art of Pure Mathematics.

Declaration

The work of this thesis is my own except where explicitly stated otherwise. The material of Chapter 1 and of Chapter 2 have been separately submitted for publication.

A note on numbering

Each chapter is divided into sections which are further divided into points. A reference to (2.3), for example, will mean the third point in section 2 of the chapter where the reference is made. If three numbers are used, for example (1.2.3) the first number denotes the chapter referred to.

Summary

This thesis is concerned with a class of flows on the 2-torus and certain properties of maps of the circle. The introduction explains the historical background to the work. In the first chapter Cherry fields are defined, and a class of natural paths introduced. It is shown that for these paths, the set of parameters for which the path takes an unstable field is a Cantor set of zero Hausdorff dimension. The Cherry fields are shown to form a family of codimension one sub-manifolds, and this is used to show that the natural paths through them are stable paths. The second chapter is concerned with rotation intervals for endomorphisms of the circle and the individual rotation intervals of different points on the circle. A shift space is constructed for each endomorphism and this is used to give new proofs of some known results and to give new information about how individual rotation intervals are distributed around the circle. The third chapter generalises a Theorem from Chapter 1 concerning rotation numbers, to one about rotation intervals.

Regarding the method of investigation, aside from the traditional methods of Pure Mathematics, some experimentation on examples was done by computer enabling the author to deduce the likely outcome of some problems. These experiments are not referred to elsewhere in the text.

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INTRODUCTION

"Up, lad: when the journey's over

There'll be time enough for sleep."

This thesis consists of three chapters. The first two may be read independently. The short third chapter presents a result using material from both the preceding chapters. Each chapter has its own introduction concerning the mathematics involved. In this general introduction we shall give some historical background and try to indicate where the work of this thesis lies in relation to the background.

In Chapter 1 we investigate an open set in the space $\mathfrak{X}^\infty(T^2)$ of smooth vector fields on the 2-torus, and certain paths (or one-parameter families) in this open set.

Recall that a vector field $X:M \to TM$ on a surface M is called Morse-Smale if it satisfies:

(MS1) All the singularities and periodic orbits are hyperbolic.

(MS2) There are no saddle connections.

(MS3) The non-wandering set of X consists of a finite number of fixed points and closed orbits.

An early landmark in the stability theory of dynamical systems is the following theorem which gives a complete description from the generic viewpoint for orientable surfaces.

Peixoto's Theorem (1962)

Let M be a compact orientable surface. The set of Morse-Smale fields is open and dense in the set of C^r vector fields on M with the C^r topology, and coincides exactly with the structurally stable fields on M .

Thus if we consider a typical path $\psi:[0,1] \to \mathcal{X}^\infty(T^2)$ we should expect that for 'most' t , $\psi(t)$ is a Morse-Smale field. It turns out that in the open set $N \subset \mathcal{X}^\infty(T^2)$ studied in Chapter 1, if we consider a natural path $\psi:[0,1] \to N$ there is a Cantor set of values of t in [0,1] for which $\psi(t)$ is not Morse-Smale. Our first result concerns the size of this Cantor set which turns out to be of zero Lebesgue measure, and so for these paths most fields are Morse-Smale also in a measure theoretic sense.

How are the unstable fields on a surface distributed amongst the Morse-Smale fields? This question was largely answered by a paper of Sotomayor [19]. We say a field <u>just fails to be (MS1)</u> if one singularity or one closed orbit is not hyperbolic. Similarly it <u>just fails to be (MS2)</u> if one pair of saddle points is connected. Define Σ_1^r = {Vector fields of class C^r on a surface which are Morse-Smale except that they just fail to be (MS1) or (MS2)}. (See [19] for a more precise definition of Σ_1^r .)

<u>Sotomayor's First Theorem</u>

1) Σ_1^r is an immersed Banach submanifold of class C^{r-1} and

codimension one in $\mathcal{X}^r(M)$.

2) Σ_1^r is dense in the unstable vector fields on M .

What about fields which are Morse-Smale except that they fail to be (MS3) ? An example of such a field is a Cherry field, described in Chapter 1. The second result of Chapter 1 is that these Cherry fields form a family of codimension one C^1 submanifolds of N . In fact in our open set $N \subset \mathcal{X}^\infty(T^2)$ every vector field which is not Morse-Smale either lies on a submanifold in Σ_1^∞ or on a submanifold of Cherry fields.

The collection of natural paths through N described above are transverse to all these submanifolds of N . The third result of Chapter 1 is that these paths are stable - that is nearby paths in the C^1 topology intersect exactly the same submanifolds of N . This fits in with another result of Sotomayor; although he cannot tell us which paths are stable, he gives a description from the generic viewpoint.

Sotomayor's Second Theorem

Let M be a compact surface and Γ the set of paths $\psi:[0,1] \to \mathcal{X}^r(M)$ which satisfy:

i) ψ takes values which are Morse-Smale or in Σ_1^r or are Morse-Smale except that they fail (MS3).

ii) ψ is transverse to Σ_1^r .

iii) The set of bifurcation values is closed and nowhere dense and its complement coincides with ψ^{-1}{structurally stable fields}.

Then Γ contains a residual subset of the space of all C^1 paths in $X^r(M)$.

To investigate flows on the torus in Chapter 1 much use is made of first return maps from the circle to itself and the most important idea is that of rotation number. In Chapter 2 we look at the generalisation of rotation numbers to non-monotonic maps, the rotation interval. This was introduced in [16] in a study of bifurcation theory and has found use, for example, in studies of transition to chaos - see [15]. We approach rotation intervals using symbolic dynamics, a method with a long and distinguished history in Dynamical Systems, used for example by Bowen in the study of Axiom A systems. The sequence space we use in fact models the dynamics very crudely but is good enough to calculate the rotation interval of a point associated to a sequence.

The final chapter finds a link between the first two by generalising a theorem in Chapter 1 about rotation numbers to one about rotation intervals. This is contrasted with a related conjecture in [16].

CHAPTER 1

The structure of the family of Cherry fields on the torus.

"About the woodlands I will go

To see the cherry hung with snow."

§1. INTRODUCTION AND STATEMENT OF RESULTS

We are interested in certain flows of class C^∞ on the 2-torus. We will work on its universal cover $\mathbb{R}^2$, so all vector fields X will satisfy $X(x+n, y+m) = X(x,y)$, for all $n,m \in \mathbb{Z}$. All the vector fields considered will satisfy the following:

(A) X has two singularities, a hyperbolic saddle S and a hyperbolic sink P.

(B) X is transverse to the circle $\Sigma = \{(x,y) | x = 0\}$.

(C) There exist $a,b \in \Sigma$ such that if $y \in (a,b)$ the positive orbit of X through y goes directly to the sink without re-intersecting Σ, but for $y \notin [a,b]$ the Poincaré map $f:\Sigma \to \Sigma$ is defined and expanding. Furthermore, $f'(y) \to \infty$ as $y \to a^-$ or $y \to b^+$. (See Figure 1a).

The Poincaré map may be extended to the whole of Σ by making it constant on [a,b] so we have a continuous circle endomorphism $f:\Sigma \to \Sigma$. By condition (C), $f'(y) \geq \lambda > 1$ for all $x \notin [a,b]$. (See Figure 1b).

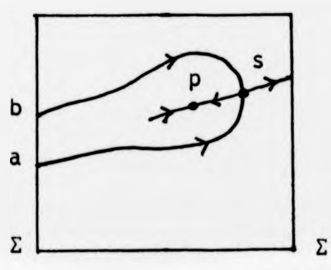


Figure 1a.

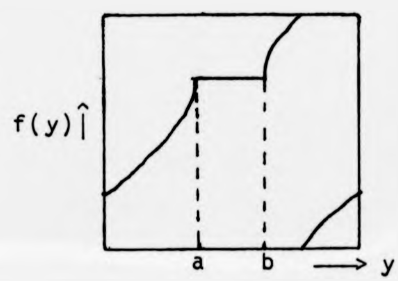


Figure 1b.

Since f is monotonic and of degree one it has a rotation number (see e.g. [17]). We denote the set of C^∞ vector fields on the 2-torus with the C^∞ topology, by $\mathfrak{X}^\infty(T^2)$, and the neighbourhood in $\mathfrak{X}^\infty(T^2)$ of all vector fields satisfying (A), (B) and (C), by N.

(1.1) Definition

A *Cherry field* is a vector field in N whose Poincaré map has irrational rotation number.

Cherry fields were first constructed in [8]; see [17], pp. 181 ff. for a modern construction. The orbit structure of a Cherry field is described by the following:

(1.2) Theorem ([17], p. 186)

Let X be a Cherry field with sink P and saddle S. Then

(1) $W^s(P)$ is dense in T^2.

(2) P and S are the only minimal sets for X.

(3) $\Sigma - W^s(P)$ is a Cantor set.

(4) $T^2 - W^s(P)$ is transitive for the flow.

Vector fields $X \in N$ whose Poincaré maps have rational rotation number are either Morse-Smale or have a saddle connection, the fields with a saddle connection forming the boundary of the Morse-Smale classes with the same rotation number. The three types of field in N are determined by what happens to the orbit of the 'free' unstable separatrix of S - that not joined directly to the sink. One of the following must happen:

(i) after intersecting Σ a number of times it intersects (a,b) and goes to the sink. In this case X is a Morse-Smale field;

(ii) after intersecting Σ a number of times it intersects Σ at a or b, so X has a saddle connection.

(iii) it intersects Σ infinitely often without intersecting $[a,b]$. In this case X is a Cherry field.

We investigate paths in N which change the relative positions of the free separatrix of $W^u(s)$ and $[a,b]$. By measuring how many parameter values correspond to Cherry fields we get an idea of how common they are in N. Let $\phi:[0,1] \to N$ be a C^1 path chosen so that $f_{\phi(t)}(y) = f_{\phi(0)}(y) + t$, where $f_{\phi(t)}$ is the Poincaré map of $\phi(t)$. Such a path may be constructed by making a suitable perturbation to the vector field in a small strip near Σ.

Every number in $[0,1)$ is represented as the rotation number of $f_{\phi(t)}$ for some t, and it is not difficult to see that the bifurcation set of ϕ is a Cantor set E. The open intervals in the complement of E consist of parameters corresponding to Morse-Smale fields, the boundary points of these intervals correspond to fields with a saddle connection and the remaining points of E correspond to Cherry fields. Our first result reveals that this path contains very few

Cherry fields. Let m denote Lebesgue measure.

(1.3) Theorem 1

Let $E = \{t \mid \phi_t$ is unstable$\}$. Then $m(E) = 0$ and furthermore E has zero Hausdorff dimension.

From well-known work of Sotomayor [19] it is known that the set of fields in N with a saddle connection form an immersed submanifold of N of class C^∞ and codimension one, or those with a particular rotation number from an embedded submanifold. We are able to show the following for Cherry fields:

(1.4) Theorem 2

The set of Cherry fields in N with a given rotation number form a codimension one embedded Banach submanifold of N of class C^1.

Note that the set of all Cherry fields is not an embedded submanifold, since there would be uncountably many components in any neighbourhood. We have no reason to believe that the submanifold is not, in fact, of class C^∞. Using Theorem 2 we prove the following about the path ϕ described above, which shows it is not a particularly special path.

(1.5) Theorem 3

The path ϕ is stable in the space of C^1 paths in N, as long as $\phi(0)$ is Morse-Smale.

ϕ is unusual as a stable path because the Cherry fields are Kupka-Smale but not Morse-Smale (see [19] p. 45).

§2. PROOF OF THEOREM 1

Theorem 1 will be implied by the slightly stronger:

(2.1) Theorem 1'

Let f be a continuous monotonic non-decreasing map of the circle of degree one satisfying

(1) f is constant on an interval $[a,b]$ and of class C^1 outside $[a,b]$.

(2) $\inf \{f'(y) | y \notin [a,b]\} = \lambda > 1$.

Let f_t be the map defined by $f_t(y) = f(y) + t$, $t \in [0,1]$. Let $E = \{t | f_t$ has irrational rotation number$\}$. Then $m(E) = 0$, where m denotes Lebesgue measure, and furthermore E has zero Hausdorff dimension.

Since the Poincaré map of any field in N satisfies the hypotheses of Theorem 1', it is clear that Theorem 1' implies Theorem 1. Note that it does not matter in Theorem 1 whether or not we include in E the parameter values corresponding to saddle connections, since there are only countably many of them. Theorem 1' shows a contrast with the situation for diffeomorphisms of the circle, by comparison with the following:

(2.2) Theorem (Herman [10])

Let f_t, $t \in [0,1]$ be a C^1 path in the space of C^r diffeomorphisms of S^1 with the C^r topology, $r \geq 3$. Let $E = \{t \mid f_t$ is C^{r-2} conjugate to an irrational rotation$\}$. As long as the rotation number changes at all along the path, then $m(E) > 0$.

(2.3) Proof of Theorem 1'

We will assume, without loss of generality, that $a = 0$ i.e. f is constant on $[0,b]$, $0 < b < 1$, and that $f(0) = 0$ (see Figure 2a). Consider the set $A \subset S^1 \times [0,1]$ defined by $A = \{(y,t) \mid y \in f_t^{-n}[0,b]$ for some $n \geq 0\}$. (See Figure 2b). We write

$$A_y = \{t \mid (y,t) \in A\} \quad A^t = \{y \mid (y,t) \in A\}.$$

Now $[0,1] - E = \{t \mid f_t$ has a periodic point$\}$

$= \{t \mid t$ is periodic$\}$ since t is always in the periodic orbit

$= \{t \mid f_t^n(b) \in [0,b]$ for some $n \geq 1\}$ since $f_t(b) = t$

$= A_b'$ where for $y \in S^1$ we write

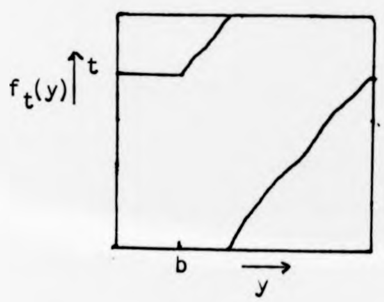


Figure 2a

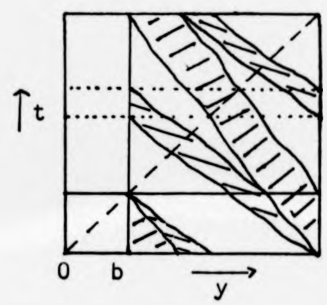


Figure 2b

$$A'_y = \{t \mid y \in f_t^{-n}[0,b] \text{ for some } n \geq 1\}.$$

Note that $A'_y = A_y$ when $y \notin [0,b]$. We will show that in fact $m(A_y) = 1$ for all $y \in S^1$. The first step is

(2.4) Lemma

$$m(A^t) = 1 \text{ for all } t \in [0,1].$$

Proof

Fix $t \in [0,1]$. We consider two distinct cases.

Case 1

$f_t^{-i}([0,b])$ intersects $f_t^{-j}[0,b]$ for some $i \neq j$. In this case f_t has a periodic point of period $n = |i-j|$. Consider the graph of f_t^n. This has n constant intervals separated by n intervals where $(f_t^n)'(y) > 1$. Hence f_t has exactly one attracting periodic orbit and one repelling periodic orbit. The attracting periodic orbit includes a point in $[0,b]$. Hence all but finitely many points end up in $[0,b]$ and so $\bigcup_{n=0}^{\infty} f_t^{-n}([0,b])$ has measure one; that is $m(A^t) = 1$.

Case 2

$f_t^{-i}[0,b] \cap f_t^{-j}[0,b]$ is empty when $i \neq j$. It follows that $|f_t^{-n}([0,b])| < b/\lambda^n \to 0$ as $n \to \infty$, since $\lambda > 1$. Now suppose for a contradiction that $m\left(\bigcup_{n=0}^{\infty} f_t^{-n}([0,b])\right) < 1$. Let us write $a_n = \sum_{i=0}^{n} |f_t^{-i}[0,b]|$. Then the monotonic sequence $\{a_n\}_{n=1}^{\infty} \to \ell$ for some $\ell < 1$. To each $f_t^{-i}[0,b]$, $i \geq 1$, there corresponds a section of the graph of f of height $|f_t^{-i-1}([0,b])|$ and length $|f_t^{-i}([0,b])|$ (see Figure 3). Chosen N so large that $\frac{1-a_N}{1-a_{N+1}} < \lambda$, which is possible since $\frac{1-a_N}{1-a_{N+1}} \to 1$ as $N \to \infty$. After removing the sections of the graph of f corresponding to $f_t^{-i}([0,b])$, $0 \leq i \leq N$, there remain at most N + 2 sections of total length $1-a_{N+1}$ and total height $1-a_N$. Since on each of these sections $f'(y) \geq \lambda$, by the Mean Value Theorem $\frac{1-a_N}{1-a_{N+1}} \geq \lambda$. This contradiction completes the proof of (2.4). □

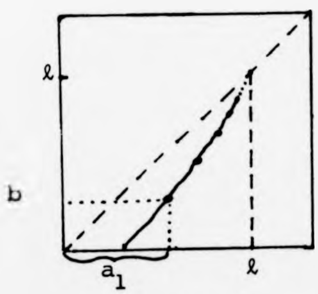


Figure 3.

(2.5) <u>Remark</u>

In particular, the Cantor set $\Sigma - W^s(P)$ mentioned in (1.2) has zero Lebesgue measure. This is because it is a set A^t

for some t in Case 2. This Cantor set also has zero Hausdorff dimension by similar arguments to those below using (2.7).

(2.6) Lemma

$m(A'_y)$ is a continuous function of y.

Proof

Consider the functions $f_{y,n}(t):[0,1] \to S^1$ defined by $f_{y,n}(t) = f_t^n(y)$, $n \geq 1$. Thus

$$f_{y,1}(t) = f(y) + t$$

$$f_{y,2}(t) = f(f_{y,1}(t)) + t$$

$$\vdots$$

$$f_{y,n}(t) = f(f_{y,n-1}(t)) + t.$$

Then $A'_y = \{t \mid f_{y,n}(t) \in [0,b] \text{ for some } n \geq 1\}$. Let us write $B_n(y) = \{t \mid f_{y,n}(t) \in [0,b] \text{ but } f_{y,m}(t) \notin [0,b] \text{ for } m < n\}$. Now $f_{y,n}(t)$ is a map of degree n. Hence B_n consists of at most n intervals each of length not more than $b/(\lambda^n+\lambda^{n-1}+\ldots+\lambda+1)$ Furthermore, since $f_{y,n}(t)$ changes continuously with y, the length of each of these intervals changes continuously with y. Hence $m(B_n(y))$ is a continuous function of y, and

$$m(B_n(y)) \leq mb/(\lambda^n + \ldots + \lambda + 1) \text{ for all } y.$$

But $A'_y = \bigcup_{n=1}^{\infty} B_n(y)$ and so $m(\bigcup_{j=1}^{n} B_j(y))$ converges uniformly to $m(A'_y)$. Hence $m(A'_y)$ is continuous as required. □

From (2.4) it follows by Fubini's Theorem (see e.g. [20], p. 143) that $m(A_y) = 1$ for almost all y. But for $y \notin [0,b]$, $m(A_y) = m(A'_y)$. Hence by (2.6) it follows that $m(A'_b) = 1$. Since $A'_b = [0,1] - E$, as noted above, we have shown $m(E) = 0$. To complete the proof we make use of

(2.7) <u>Proposition</u> (Besicovitch and Taylor [4])

Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n = 1$. Let $E \subset [0,1]$ be a set whose complement is a union of intervals A_n with $m(A_n) = a_n$. Then
$\dim_H(E) < \inf \{\beta \mid \sum_{n=1}^{\infty} a_n^{\beta} < \infty\}$.

Here $\dim_H(E)$ denotes the Hausdorff dimension of E. (See [12] for the definition). For any positive integer n let $\phi(n)$ be the number of positive integers coprime to n and less than n. Then as in the proof of (2.6) we see that the complement of E, A'_b, consists of $\phi(n)$ intervals for each n, each of length not more than $b/(\lambda^n + \ldots + \lambda + 1) = \dfrac{b(\lambda-1)}{(\lambda^{n-1}-1)}$. Now

$(\lambda-1) \sum_{n=1}^{\infty} \dfrac{\phi(n)}{(\lambda^{n+1}-1)^{\beta}} < \infty$ for all $\beta > 0$. Hence by (2.7) we have $\dim_H(E) = 0$. The proof of Theorem 1' is complete. □

(2.8) Example

Consider the 2-parameter family of endomorphisms of the circle defined by

$$f_{a,t}(x) = \begin{cases} t & \text{if } x < a \\ \dfrac{x-a}{1-a} + t & \text{if } x > a \end{cases}$$

for $x \in S^1$, $t \in [0,1]$ and $0 < a < 1$. (See Figure 4). For any fixed a, $f_{a,0}$ satisfies the hypothesis of Theorem 1'. In the same way as Arnold and Herman do for diffeomorphisms ([2], p.273, [10], p.280) we can consider level sets for the rotation number. To each rational number a "balloon" is attached, being the level set for that rotation number.

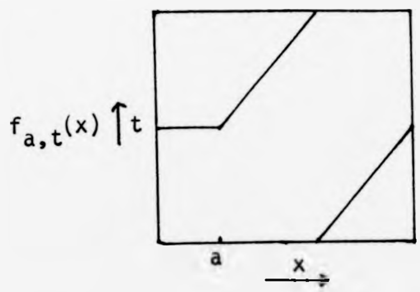


Figure 4.

(These balloons are analogous to the so-called 'Arnold tongues'.) Even though the width of each balloon tends to zero as a does, Theorem 1' tells us that for each $a_0 > 0$, the line at height a_0 intersects the balloons in a set of measure one (See Figure 5).

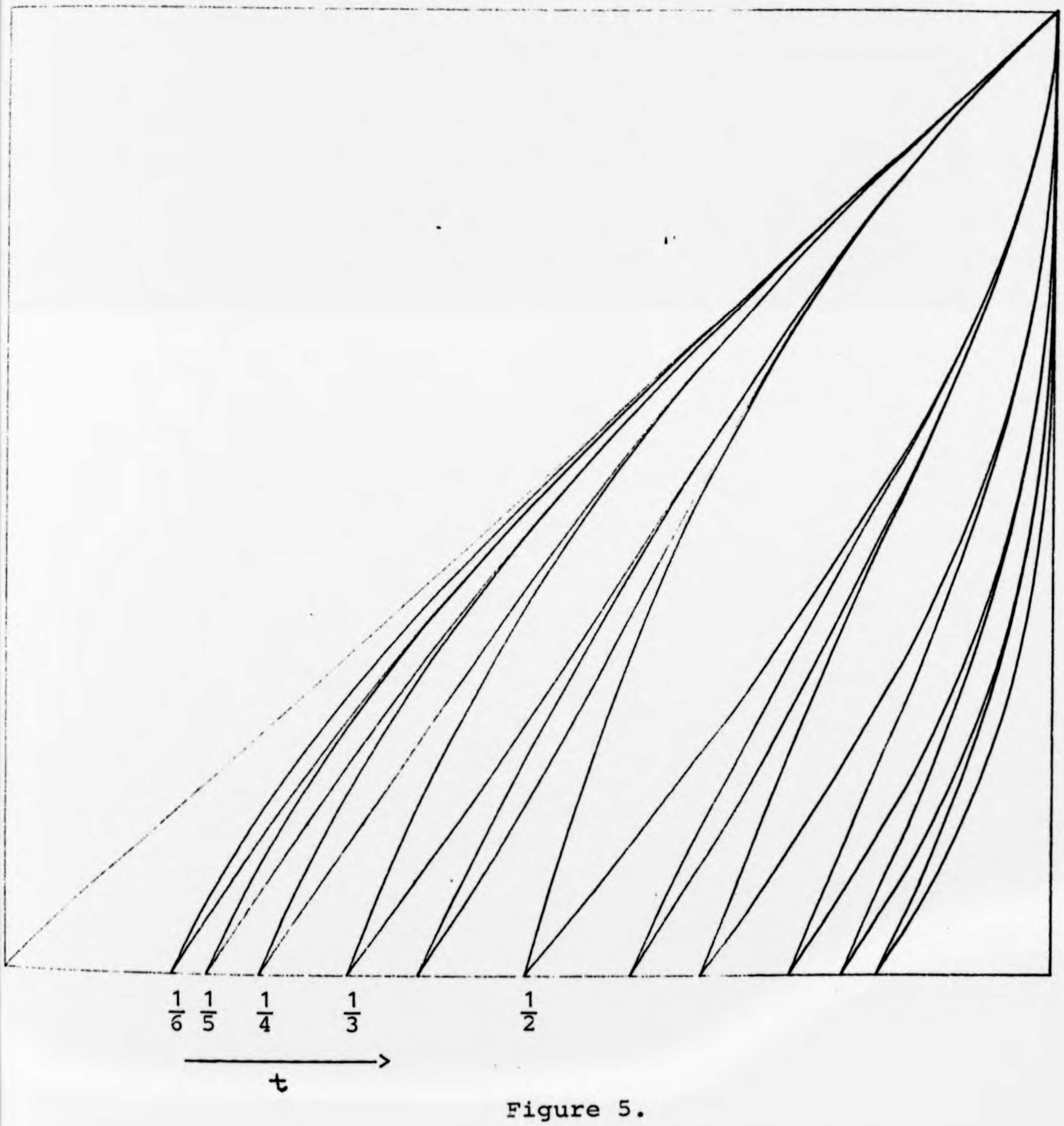


Figure 5.

§3. PROOF OF THEOREM 2

We turn to the proof of Theorem 2. From now on we will write f_Y for the Poincaré map of a vector field $Y \in N$, and $\rho(f_Y)$ for its rotation number. As in [19], the procedure is to construct, for each irrational $\alpha \in [0,1)$, a C^1 function $g_\alpha : N \to \mathbb{R}$ such that $g_\alpha^{-1}(0) = \{Y \in N | \rho(f_Y) = \alpha\}$ and $Dg_\alpha(Y) \neq 0$. (Strictly, we will choose g_α to have image S^1). We first do this for rational rotation numbers m/n, using the Implicit Function Theorem to construct g_n, h_n (for notational convenience we suppress the m's) which take respectively the lower and upper boundaries of the Morse-Smale class $\{Y | \rho(f_Y) = m/n\}$ onto zero (see Figure 6). Then g_α is shown to be the C^1 limit of g_{n_j} or h_{n_j} when $(m_j/n_j) \to \alpha$. More precisely we show that for any $Y \in N$ and a Cauchy

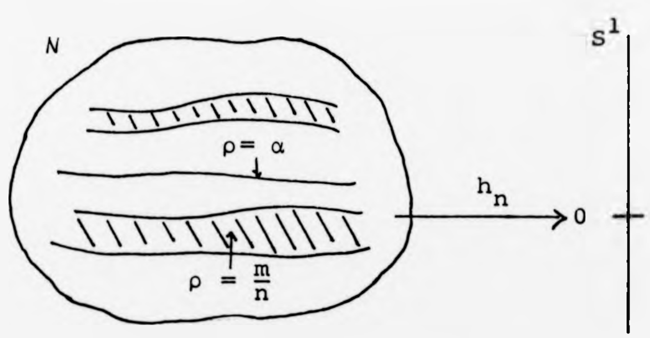


Figure 6.

sequence $(m_j/n_j)_{j=1}^{\infty}$, the sequence $(Dg_{n_j}(Z))_{j=1}^{\infty}$ is uniformly Cauchy for Z in a neighbourhood of Y. This also shows that any two of these manifolds that are close are in fact C^1-close, which is crucial for Theorem 3. It turns out that $g_{\alpha}(Y)$ will be the solution for t of $\rho(f_Y + t) = \alpha$; that is how far the graph of f_Y must be lifted to have rotation number α.

Let $[a_Y, b_Y]$ be the interval on which f_Y is constant, and let $f_Y(y) = t_Y$ for all $y \in [a_Y, b_Y]$. We may consider a, b and t to be functions of Y, $N \to S^1$ and by the Stable Manifold Theorem (see e.g. [11]) they are of class C^{∞} (consider Figure 1). We will write $f_Y + t$ for the function defined by $(f_Y + t)(x) = f_Y(x) + t$. Then $f_Y + t$ has a point of period n exactly when $(f_Y + t)^{n-1}(t_Y + t) \in [a_Y, b_Y]$. Define $\psi_n : N \times S^1 \to S^1$ by

$$\psi_n(Y,t) = (f_Y + t)^{n-1}(t_Y + t).$$

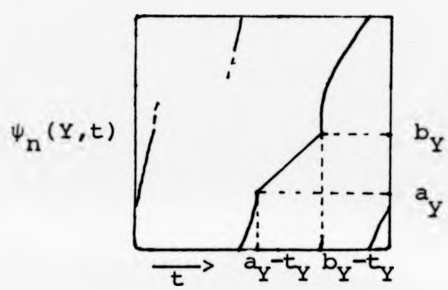


Figure 7

So $\psi_n(Y,\cdot)$ is a circle map of degree n. The boundary points of the Morse-Smale class with rotation number m/n are given by the m'th solutions of $\psi_n(Y,t) = a_Y$ or b_Y where the solutions are counted with increasing t, $a_Y - t_Y$ and $b_Y - t_Y$ being the zero-th.

Now in a neighbourhood of such a solution (Y,t_0), ψ_n is of class C^∞. This follows since outside $[a_Y, b_Y]$, f_Y is a 'genuine' Poincaré map and so the map $(Y,t) \longmapsto f_Y(t)$ is of class C^∞ (see [19] p.9). Also it is clear that $D_2\psi_n(Y,t_0) > 0$. Hence we may apply the Implicit Function Theorem ([9] p. 270). This tells us that there is a neighbourhood $\mathcal{B}$ of Y in N and C^∞ functions g_n, $h_n : \mathcal{B} \to S^1$ satisfying

$$\psi_n(Y, g_n(Y)) - a(Y) = 0$$

$$\psi_n(Y, h_n(Y)) - b(Y) = 0$$

and hence

$$Dg_n(Y) = -\frac{D_1\psi_n(Y,g_n(Y)) - Da(Y)}{D_2\psi_n(Y,g_n(Y))} . \tag{3.1}$$

We may write it in this form since $D_2\psi_n(\, , \,)$ is a real number. Obviously we have the same formula for Dh_n with $Da(Y)$ replaced by $Db(Y)$.

Thus $g_n(Y) - a(Y)$ is a C^∞ function of Y taking the saddle connection fields with rotation number m/n to zero in S^1.

Note that $D(g_n(Y) - a(Y)) \neq 0$, since, for example, it is non-zero on our particular path ϕ in §2. This shows that these vector fields with saddle connections form a C^∞ codimension one submanifold of N, as is known from [19]. Note also that $g_n(Y) - a(Y)$ is defined for all $Y \in N$ and C^∞ everywhere.

From now on fix $Y \in N$ and choose a Cauchy sequence of rationals $\{m_j/n_j\}_{j=1}^{\infty}$. To prove Theorem 2 it is sufficient to show that $\{Dg_{n_j}(Z)\}_{j=1}^{\infty}$ is uniformly Cauchy for Z in a neighbourhood of Y as $j \to \infty$ (see [9] p. 163). It will be clear from the proof that we could have allowed any Cauchy sequence of saddle connection fields and their corresponding functions h_{n_j} or g_{n_j}. For simplicity we just consider the functions g_{n_j}.

We may write f_Y as a map of two variables: $f(Y,t) = f_Y(t)$. We rewrite Dg_n in terms of D_1f and D_2f. For $n > 2$, let $\mu(Y,t,s) = f_Y(t) + s$ and $\nu(Y,t) = (Y,\psi_{n-1}(Y,t),t)$. Then $\psi_n = \mu\circ\nu$. Hence, by the Chain rule,

$$D\psi_n(Y,t) = (D_1f(\psi_{n-1}(Y,t)) + D_2f(\psi_{n-1}(Y,t)).D_1\psi_{n-1}(Y,t)$$

$$D_2f(\psi_{n-1}(Y,t)).D_2\psi_{n-1}(Y,t)+1).$$

Hence from (3.1) and by induction

(3.2) $$Dg_n(Y) = -\frac{\alpha_{n,n}+\beta_{n,n}(\alpha_{n,n-1}+\beta_{n,n-1}(\alpha_{n,n-2}\ldots(\alpha_{n,2}+\beta_{n,2}.Dt(Y))..)-Da(Y}{\beta_{n,n}(\beta_{n,n-1}(\ldots(\beta_{n,2}+1)+1)\ldots)+1}$$

where $\alpha_{n,j} = D_1f(Y,x_{j-1})$ and $\beta_{n,j} = D_2f(Y,x_{j-1})$ and here $x_j = (f_Y + g_n(Y))^{j-1}(t_Y + g_n(Y))$ - that is the (j-1) st. iterate of $t_Y + g_n(Y)$ which is in the periodic orbit for $f_Y + g_n(Y)$. Since the $\beta_{n,j}$'s are real numbers, (3.2) is the same as

(3.3) $$Dg_n(Y) = \frac{-\alpha_{n,n}}{\beta_{n,n}(\beta_{n,n-1}(\ldots)+1)+1} + \frac{-\alpha_{n,n-1}}{\beta_{n,n-1}(\ldots)+1+1/\beta_{n,n}} + \ldots$$

$$\ldots + \frac{-Dt(Y)}{1+\frac{1}{\beta_{n,2}}+\ldots+\frac{1}{\beta_{n,2}\ldots\beta_{n,n}}} + \frac{Da(Y)}{\beta_{n,n}(\beta_{n,n-1}(\ldots)+1)+1}$$

So to calculate Dg_n we need to know the values of $D_1f(Y,x_j)$ and $D_2f(Y,x_j)$ for x_j in the periodic orbit of $f_Y + g_n(Y)$. To deal with the case when this orbit comes close to a_Y or b_Y we need

(3.4) <u>Proposition</u>

$$-\frac{D_1f(Z,a_z-\delta)}{D_2f(Z,a_z-\delta)} \longrightarrow Da(Z) \text{ as } \delta \to 0^+$$

$$-\frac{D_1f(Z,b_Y+\delta)}{D_2f(Z,b_Y+\delta)} \longrightarrow Db(Z) \text{ as } \delta \to 0^+$$

(3.5) Corollary

$$\frac{||D_1 f(Z,x)||}{D_2 f(Z,x)}$$

is uniformly bounded for $x \notin [a_z, b_z]$ and Z in a small neighbourhood of Y.

Since the proof of (3.4) is quite technical, we defer it to the next section.

Fix $\gamma > 0$. We will find η so small that for Z in a small neighbourhood of Y, $|g_s(Z) - g_{\bar{s}}(Z)| < \eta \Rightarrow ||Dg_s(Z) - Dg_{\bar{s}}(Z)|| < \gamma$ where r/s and $\bar{r}/\bar{s}$ are elements of $\{m_j/n_j\}_{j=1}^{\infty}$. The first step is to show that we can ignore all but finitely many terms in (3.3).

We write its j-th term

$$T_n^j = \frac{-\alpha_{n,j}}{\beta_{n,j}(\beta_{n,j-1}(\ldots)+1)+1 + \frac{1}{\beta_{n,j+1}} + \ldots + \frac{1}{\beta_{n,j+1}\cdots\beta_{n,n}}} .$$

This is defined for $1 < j < n$ by letting $\alpha_{n,1} = Dt(Y)$.

Then if M is the uniform bound on $\dfrac{||D_1 f(Z,x)||}{D_2 f(Z,x)}$ for Z in a neighbourhood A of Y, guaranteed by (3.5) we have $||T_n^j|| < \dfrac{M}{\lambda^{j-2}}$. So we may choose n_0 so large that

(3.6) $$\sum_{j=n_0}^{\infty} \frac{M}{\lambda^{j-2}} < \frac{\gamma}{8}$$ and also

(3.7) $$\frac{||Da(Z)||}{\lambda^{n_0}} < \frac{\gamma}{8} \quad \text{and} \quad \frac{||Db(Z)||}{\lambda^{n_0}} < \frac{\gamma}{8}$$

for all $Z \in A$, if A is small enough. Next we choose δ_1 so small that if x_j comes within δ_1 of a_z or b_z then $\beta_{n,j}$ is large enough so that we may ignore T_n^i for $i > j$. Precisely, choose $\delta_1 > 0$ so that for all $Z \in A$ and $x \in (a_z-\delta_1, a_z)$ or $x \in (b_z, b_z + \delta_1)$ the following hold:

$$(3.8) \qquad \frac{1}{D_2 f(Z,x)} \sum_{j=2}^{\infty} \frac{M}{\lambda^{j-2}} < \frac{\gamma}{8}.$$

$$(3.9) \qquad \frac{\|Da(Z)\|}{D_2 f(Z,x)} < \frac{\gamma}{8} \quad \text{and} \quad \frac{\|Db(Z)\|}{D_2 f(Z,x)} < \frac{\gamma}{8}.$$

$$(3.10) \qquad \left\| \frac{D_1 f(Z,x)}{D_2 f(Z,x)} + Da(Z) \right\| < \frac{\gamma}{8} \text{ if } x \in (a_z-\delta_1, a_2) \text{ and}$$

$$\left\| \frac{D_1 f(Z,x)}{D_2 f(Z,x)} + Db(Z) \right\| < \frac{\gamma}{8} \text{ if } x \in (b_z, b_z + \delta_1).$$

$$(3.11) \qquad \frac{\lambda}{(\lambda-1)} \frac{1}{D_2 f(Z,x)} < \frac{\gamma}{8.M.n_0}.$$

The inequalities (3.10) are possible by using (3.4).

Now choose $0 < \delta_2 < \delta_1$. Let us write

$$x_j = (f_z + g_s(Z))^{j-1}(t_z + g_s(Z))$$

$$y_j = (f_z + g_{\bar{s}}(Z))^{j-1}(t_z + g_{\bar{s}}(Z)).$$

The idea is to choose η so small that $|g_s(Z) - g_{\bar{s}}(Z)| < \eta$ implies that if x_j is δ_2-close to a_z or b_z then y_j is δ_1-close to a_z or b_z. We may choose η small enough so that if

$|g_s(Z) - g_{\bar{s}}(Z)| < \eta$ then the following holds:

(3.12) if x_j and y_j are inside $[b_z + \delta_2, a_z - \delta_2]$ then

$$||T_s^j - T_{\bar{s}}^j|| \le \gamma/4n_0 \quad \text{for} \quad 1 \le j \le n_0 .$$

Note that for fixed j, to make $||T_s^j - T_{\bar{s}}^j|| < \gamma/4n_0$, it is necessary only to make $\beta_{s,k}$ and $\beta_{\bar{s},k}$ close for finitely many k, say $N(j)$. Therefore we may choose n_1 so that if $\beta_{s,k}$ and $\beta_{\bar{s},k}$ are close for $1 \le k \le n_1$ then $||T_s^j - T_{\bar{s}}^j|| < \gamma/4n_0$ for $0 \le j \le n_0$, where $n_1 = \sup\{N(j) \mid 1 \le j \le n_0\}$.

To take care of the case $s < n_1$ we also need

(3.13) if $K = \sup_{\substack{x \in [b_z + \delta_2, a_z - \delta_2] \\ Z \in A}} |D_2 f(Z,x)|$ then

$$\delta_2 - \delta_1 > \eta . K^{n_2 - 1} \quad \text{where} \quad n_2 = \max(n_0, n_1) .$$

This tells us that if $x_j \notin [b_z + \delta_2, a_z - \delta_2]$ then $y_j \notin [b_z + \delta_1, a_z - \delta_1]$, for $1 \le j \le n_2$. So using (3.11), the worst possible case for (3.12) is

$$||T_s^j - T_{\bar{s}}^j|| \le || \frac{\alpha_{s,j}}{\beta_{s,j}(\beta_{s,j-1}(\ldots)+1)+1} - \frac{\alpha_{\bar{s},j}}{\beta_{\bar{s},j}(\ \bar{s},j-1(\ldots)+1)+1+\gamma/8Mn_0} ||$$

$$\le || \frac{(\beta_{\bar{s},j}(\beta_{\bar{s},j-1}(\ldots)+1)\alpha_{s,j} - (\beta_{s,j}(\ldots)+1)\alpha_{s,j})}{(\beta_{s,j}(\ldots)+1)\ (\beta_{\bar{s},j}(\ldots)+1+\gamma/8Mn_0)} ||$$

$$+ || \frac{\alpha_{s,j}\gamma/8Mn_0}{(\beta_{s,j}(\ldots)+1)\ (\beta_{\bar{s},j}(\ldots)+1+\gamma/8Mn_0} ||$$

$\le \gamma/4n_0$ if $\beta_{s,k}$ and $\beta_{\bar{s},k}$ are close for $1 \le k \le j$ and $\alpha_{s,j}$ and $\alpha_{\bar{s},j}$ are close, which is true if $|g_s(Z) - g_{\bar{s}}(Z)|$ is small enough.

In a similar way we can ensure that if x_j and y_j are both in $(a_z - \delta_1, a_z]$ or $[b_z, b_z + \delta_1)$ then for $|g_s(Z) - g_{\bar{s}}(Z)| < \eta$ the following holds:

$$(3.14)\quad ||Da(Z)|| \left(\frac{1}{\beta_{s,j}(\ldots)+1+1/\beta_{s,j+1} + \ldots + 1/\beta_{s,j+1}\cdots\beta_{s,s}} - \frac{1}{\beta_{\bar{s},j}(\ldots)+1+1/\beta_{\bar{s},j+1} + \ldots + 1/\beta_{\bar{s},j+1}\cdots\beta_{\bar{s},\bar{s}}} \right) < \frac{\gamma}{8}$$

for $1 \le j \le n_0$ and $Z \in A$.

We now claim that

$$|g_s(Z) - g_{\bar{s}}(Z)| < \eta \Rightarrow ||Dg_s(Z) - Dg_{\bar{s}}(Z)|| < \gamma\ .$$

We consider three cases:

Case 1

x_j and y_j are in $[b_z + \delta_2, a_z - \delta_2]$ for $1 < j < n_0$. Consider the equation (3.3). The terms T^j_s, $T^j_{\bar{s}}$ are taken care of, for $j > n_0$, by (3.6), and each final term by (3.7). The other terms are dealt with by (3.12). Thus

$$\|Dg_s(Z) - Dg_{\bar{s}}(Z)\| < \frac{\gamma}{4} + \frac{\gamma}{8} + \frac{\gamma}{8} + \frac{\gamma}{8} + \frac{\gamma}{8} < \gamma .$$

If case 1 does not happen let j_0 be the first j where it fails. Suppose $x_{j_0} \notin [b_2 + \delta_2, a_z - \delta_2]$. Clearly the case $y_{j_0} \notin [b_z + \delta_2, a_z - \delta_2]$ is similar.

Case 2

$x_{j_0} = a_z$. This is the case $s = j_0$, since there is a point of period j_0. If also $y_{j_0} = a_z$ then Dg_s and $Dg_{\bar{s}}$ differ only in their final terms and this case is clearly alright. Otherwise, by (3.13), $y_{j_0} \in (a_z - \delta_1, a_z)$. For $j < j_0$ the terms T^j_s, $T^j_{\bar{s}}$ are dealt with by (3.12) and for $j > j_0 + 1$, $T^j_{\bar{s}}$ are dealt with by (3.8). Hence

$$\|Dg_s(z)-Dg_{\bar{s}}(Z)\| < \frac{j_0\cdot\gamma}{4n_0} + \frac{\gamma}{8} + \left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s-1}(\ldots)+1)+1} - T_{\bar{s}}^{j_0+1}\right\|$$

$$+ \frac{\|Da(Z)\|}{\beta_{\bar{s},\bar{s}}(\beta_{\bar{s},\bar{s}-1}(\ldots)+1)+1}$$

Since $y_{j_0} \in (a_z-\delta_1, a_z)$ it follows by (3.10) that

$$\left\|\frac{D_1f(Z,y_{j_0})}{D_2f(Z,y_{j_0})} + Da(z)\right\| < \frac{\gamma}{8} \text{ or } \left\|\frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}} + Da(Z)\right\| < \frac{\gamma}{8}\ .$$

Thus

$$\left\|T_{\bar{s}}^{j_0+1} - \frac{Da(Z)}{\beta_{\bar{s},j_0}(\ldots)+1+\ldots+1/\beta_{\bar{s},j_0+1}\ldots\beta_{\bar{s},\bar{s}}}\right\| < \frac{\gamma}{8}$$

and so

$$\left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s,-1}(\ldots)+1)+1} - T_{\bar{s}}^{j_0+1}\right\| < \frac{\gamma}{8} + \left\|\frac{Da(Z)}{\beta_{s,s}(\beta_{s,s-1}(\ldots)+1)+1} - \right.$$

$$\left. - \frac{Da(Z)}{\beta_{\bar{s},j_0}(\ldots)+1+\ldots+1/\beta_{\bar{s},j_0+1}\ldots\beta_{\bar{s},\bar{s}}}\right\| < \frac{\gamma}{8} + \frac{\gamma}{8} \text{ by (3.14).}$$

Finally, by (3.7) $\dfrac{\|Da(Z)\|}{\beta_{\bar{s},\bar{s}}(\beta_{\bar{s},\bar{s}-1}(\ldots)+1)+1} < \dfrac{\gamma}{8}$.

Thus $\|Dg_s(Z) - Dg_{\bar{s}}(Z)\| < \frac{\gamma}{4} + \frac{\gamma}{8} + \frac{\gamma}{3} + \frac{\gamma}{8} < \gamma$.

Case 3

$x_{j_0} \in (a_z - \delta_2, a_z)$. Clearly the case $x_{j_0} \in (b_z, b_z + \delta_2)$ is similar. By (3.13), $y_{j_0} \in (a_z-\delta_1, a_z)$. Using (3.12) and (3.8) as in Case 2, we have

$$\|Dg_s(Z) - Dg_{\bar{s}}(Z)\| < \frac{j_0 \cdot \gamma}{4n_0} + \frac{\gamma}{8} + \frac{\gamma}{8} +$$

$$\left\| \frac{\alpha_{s,j_0+1}}{\beta_{s,j_0+1}(\ldots)+1+\ldots+1/\beta_{s,j_0}\ldots\beta_{s,s}} - \frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}(\ldots)+1+\ldots+1/\beta_{\bar{s},j_0},\ldots\beta_{\bar{s},\bar{s}}} \right\| +$$

$$\|Da(Z)\| \left(\frac{1}{\beta_{s,s}(\ldots) + 1} - \frac{1}{\beta_{\bar{s},\bar{s}}(\ldots) + 1} \right)$$

Since both x_{j_0} and y_{j_0} are in $(a_z-\delta_1, a_z)$ it follows from (3.10) that

$$\left\| \frac{\alpha_{s,j_0+1}}{\beta_{s,j_0+1}(\ldots)+1+\ldots+1/\beta_{s,j_0+2}\ldots\beta_{s,s}} - \frac{\alpha_{\bar{s},j_0+1}}{\beta_{\bar{s},j_0+1}(\ldots)+1+\ldots+1/\beta_{\bar{s},j_0+2}\ldots\beta_{\bar{s},\bar{s}}} \right\|$$

$$< \frac{\gamma}{8} + \frac{\gamma}{8} + \|Da(Z)\| \left(\frac{1}{\beta_{s,j_0}(\ldots)+1+\ldots+1/\beta_{s,j_0+1}\ldots\beta_{s,s}} \right.$$

$$\left. - \frac{1}{\beta_{\bar{s},j_0}(\ldots)+1+\ldots+1/\beta_{\bar{s},j_0+1}\ldots\beta_{\bar{s},\bar{s}}} \right) < \frac{\gamma}{4} + \frac{\gamma}{8} \text{ by (3.14).}$$

By applying (3.7) to the final term we now have

$$\|Dg_s(Z) - Dg_{\bar{s}}(Z)\| < \frac{\gamma}{2} + \frac{\gamma}{4} + \frac{\gamma}{8} + \frac{\gamma}{8} = \gamma.$$

Thus Theorem 2 is proved. □

§4. PROOF OF (3.4) AND (3.5)

The idea of the proof of (3.4) is to show that near to the saddle separatrix $f(Z,x)$ behaves like the Poincaré map of a linear saddle. But for a linear vector field L with matrix $\begin{pmatrix}\lambda(L) & 0 \\ 0 & -\mu(L)\end{pmatrix}$ its Poincaré map $p(L,x)$ behaves like $x \longmapsto x^{\mu(L)/\lambda(L)}$

Hence $D_1p(L,x)/D_2p(L,x)$ behaves like

$$\frac{D(\mu(L)/\lambda(L)).x^{\mu(L)/\lambda(L)}\log x}{x^{\mu(L)/\lambda(L)}} = D(\mu(L)/\lambda(L)).x \log x \to 0$$

as $x \to 0$. Because the saddle point of Z moves with Z the term $Da(Z)$ or $Db(Z)$ enters as a correction term.

Let us call the saddle point of Z, $S(Z)$. By Sell's Linearization Theorem [18] there is a C^1 map $\ell_z : U \to \mathbb{R}^2$ from some neighbourhood U of $S(Z)$ conjugating the flow to its linear part $L_z = DZ_{s(z)}$:

$$\ell_z(\psi_z(x,t)) = \psi_{L_z}(\ell z(x),t)$$

if x and $\psi_z(x,t) \in U$. (Here ψ (,) denotes the flow induced by the vector field X.) Furthermore, Sell shows that the linearization can be chosen to depend in a C^1 way upon Z([18] p. 64), so $\ell(Z,t)$ is a C^1 map. Therefore we choose $\ell(Z,t)$ first for $Z = Y$. Then the neighbourhood referred to in (3.5) in which Z is allowed to lie is the domain of definition of the first component of ℓ. Call this neighbourhood $\mathcal{B}$. For points inside $\bigcup_{t\in\mathbb{R}} \psi_z(t,U)$ we can extend ℓ_z, so long as its domain of definition does not overlap itself, by setting

$$\ell_z(x) = \psi_{L_z}(\ell_z(\psi_z(x,t),-t)$$

where t is chosen so that $\psi_z(x,t) \in U$ and so that the partial orbit joining x to $\psi_z(x,t)$ has not passed through U. Hence we may extend the domain of ℓ_z so far as to include intervals in Σ, $(a_z-\beta,a_z)$ and $(t_z-\gamma,t_z)$ for small β,γ. (See Figure 8a). Note that $\ell(Z,t)$ is still a C^1 map on this extended domain. Now for $t \in (a_z-\beta,a_z)$ let

$$\ell(Z,t) = \begin{pmatrix} x(Z,t) \\ \sigma(x(Z,t)) \end{pmatrix} \text{ and for } t \in (t_z-\gamma,t_z) \text{ let}$$

$$\ell(Z,t) = \begin{pmatrix} \tau(y(Z,t)) \\ y(Z,t) \end{pmatrix}$$

where $\sigma:(0,\bar{\beta}) \to R$ and $\tau:(0,\bar{\gamma}) \to R$ are C^1 functions for some small $\bar{\beta}$ and $\bar{\gamma}$. (See Figure 8b.) So for $t \in (a_z-\beta,a_z)$, $f_z(t)$ progresses thus:

$$t \xmapsto{\ell_z} \begin{pmatrix} x(Z,t) \\ \sigma(x(Z,t)) \end{pmatrix} \xmapsto{p_z} \begin{pmatrix} \tau_z(p_z(x(z,t))) \\ p_z(x(Z,t)) \end{pmatrix} \xmapsto{\ell_z-1} f_z(t)$$

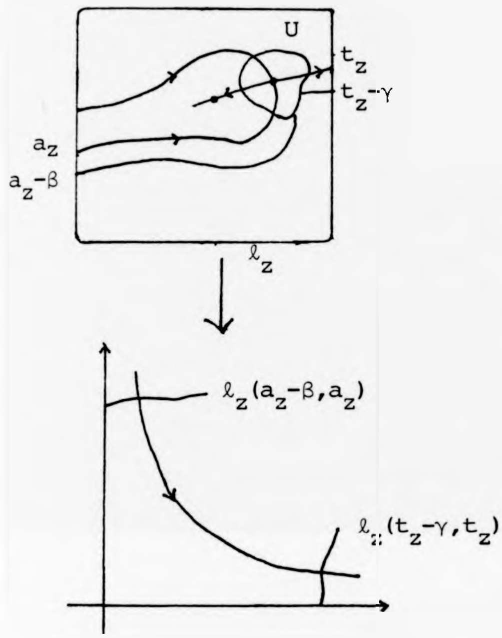


Figure 8a

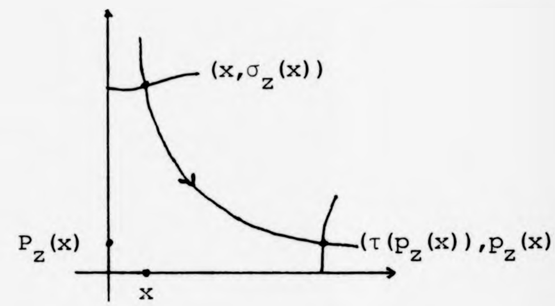


Figure 8b

where $p_z:[0,\bar{\beta}) \to [0,\bar{\gamma})$ is the Poincaré map of L_z from $\ell_z(a_z-\beta,0)$ to $\ell_z(t_z-\gamma,0)$.

Let $\Pi_x:\ell_z(a_z-\beta,a_z] \to \mathbb{R}$ and $\Pi_y:\ell_z(t_z-\gamma,t_z) \to \mathbb{R}$ be the projections: $\Pi_x\begin{pmatrix} x \\ \sigma_z(x)\end{pmatrix} = x$ and $\Pi_y\begin{pmatrix}\tau_z(y) \\ y\end{pmatrix} = y$.

Then if we set

$$j(Z,t) = \Pi x \circ \ell(Z,t)$$
$$k(Z,t) = \ell_z^{-1} \circ \Pi_y^{-1}(Z,t) \quad \text{we have}$$

$$(4.1)\qquad f(Z,t) = k(Z,p_Z(j(Z,t))).$$

Let us write $F_\delta(Z) = f(Z,a_Z-\delta)$. Then

$$(4.2)\qquad DF_\delta(Z) = D_1 f(Z,a_Z-\delta) + D_2 f(Z,a_Z-\delta).Da(Z).$$

On the other hand we also have

$$F_\delta(Z) = k(Z,p_Z(j(Z,a_Z-\delta)))$$

and so

$$(4.3)\quad DF_\delta(Z) = D_1 k + D_2 k.D_1 p + D_2 k.D_2 p.D_1 j + D_2 k.D_2 p.D_2 j.Da(Z).$$

(In order to simplify the equations, from now on we are omitting the points at which Dk, Dj and Dp are evaluated). From (4.1) we also have

$$(4.4)\qquad D_2 f(Z,t) = D_2 k.D_2 p.D_2 j.$$

Together (4.2), (4.3) and (4.4) give us

$$(4.5)\quad \frac{D_1 f(Z,a_Z-\delta)}{D_2 f(Z,a_Z-\delta)} = \frac{D_1 k}{D_2 k.D_2 p.D_2 j} + \frac{D_1 p}{D_2 p.D_2 j} + \frac{D_1 j}{D_2 j} .$$

We consider each term on the right-hand side separately.

$$\lim_{\delta\to 0} \frac{D_1 k}{D_2 k.D_2 p.D_2 j} = \frac{D_1 k(Z,0)}{D_2 k(Z,0) D_2 j(Z,0)} . \lim_{\delta\to 0} \frac{1}{D_2 p}$$

$$= 0 \text{ as we will see below.}$$

Note that this convergence is uniform in $\mathcal{B}$ since j and k are bounded in $\mathcal{B}$.

$$\lim_{\delta\to 0} \frac{D_1 p(Z,j(Z,a_z-\delta))}{D_2 p(Z,j(Z,a_z-\delta))} = \lim_{x\to 0} \frac{D_1 p(Z,x)}{D_2 p(Z,x)} \cdot \frac{1}{D_2 j(Z,a_z)} .$$

We consider this case below. For the first term let us write $J(Z,\delta) = j(Z,a_z-\delta)$. Then

$$\frac{D_1 J(Z,\delta)}{D_2 J(Z,a_z-\delta)} = \frac{D_1 j}{D_2 j} + Da(Z).$$

Since $D_1 J(Z,0) = 0$ and J is C^1, we deduce

$$\lim_{\delta\to 0} \frac{D_1 j}{D_2 j} = - Da(Z).$$

Thus from (4.5) we now have

$$(4.6) \quad \lim_{\delta\to 0} \frac{D_1 f(Z,a_z-\delta)}{D_2 f(Z,a_z-\delta)} = -Da(Z) + \frac{1}{D_2 j(Z,a_z)} \lim_{x\to 0} \frac{D_1 p(Z,x)}{D_2 p(Z,x)} .$$

Hence it now suffices to examine $p(Z,x)$, the Poincare map of L_z. We may assume L_z has the form $\begin{pmatrix} \lambda(Z) & 0 \\ 0 & -\mu(Z) \end{pmatrix}$. Let us write

$\alpha(Z) = \mu(Z)/\lambda(Z)$. Then we may integrate the vector field L_z to find

$$(4.7) \qquad p(Z,x) = \sigma_z(x) \left(\frac{x}{\tau_z(p_z(x))}\right)^{\alpha(z)}$$

Differentiating we get

$$D_1p(Z,x) = x^{\alpha(z)}\left[\frac{D_1\sigma.\tau+\sigma.D\alpha[\log x-\log \tau]\tau-\sigma.\alpha.D_1\tau}{\tau^{\alpha(z)+1} + D_2\tau.\sigma.x^{\alpha(z)}}\right]$$

$$D_2p(Z,x) = x^{\alpha(z)-1}\left[\frac{D_2\sigma.\tau.x + \sigma.\tau.\alpha}{\tau^{\alpha(z)+1}+\sigma.\alpha.x^{\alpha(z)}.D_2\tau}\right]$$

and hence that $\lim\limits_{x\to 0} \dfrac{D_1p(Z,x)}{D_2p(Z,x)} =$

$$\lim_{x\to 0} \frac{x[(D_1\sigma.\tau + \sigma.D\alpha[\log x-\log \tau]\tau - \sigma.\alpha.D_1\tau).\tau^{\alpha+1}]}{\tau^{\alpha+1}.\sigma.\tau.\alpha}$$

$$= 0 \text{ since } \lim_{x\to 0} x \log x = 0.$$

Notice that since σ, τ and α are bounded in B this convergence is uniform in B. By (4.6) the proof of (3.4) is complete.

<u>Proof of (3.5)</u>

Consider (4.5). We have already noted that the first two terms on the right hand side are bounded uniformly in B. But the final term is defined and continuous for $\delta = 0$ so is also bounded in B. Thus (3.5) is proven.

§5. PROOF OF THEOREM 3

Let $C^1(I,\mathfrak{X})$ be the space of C^1 paths in $\mathfrak{X}^\infty(T^2)$ with the C^1 topology. We recall what it means for a path to be stable (see e.g. [21]).

(5.1) Definition

Two paths $X,Y \in C^1(I,\mathfrak{X})$ are (*mildly*) *equivalent* if there exists a reparametrising homeomorphism $h:I \to I$ such that $X(t)$ and $Y(h(t))$ are topologically equivalent vector fields. X and Y are *strongly equivalent* if in addition the topological equivalence between $X(t)$ and $Y(h(t))$ can be chosen to change continuously with t. X is a *stable path* if there is a neighbourhood U of X in $C^1(I,\mathfrak{X})$ such that all paths Y in U are strongly equivalent to X.

Before showing that our chosen path ϕ is stable we note that all Cherry fields with the same rotation number are topologically equivalent. Similarly all Morse-Smale fields in the neighbourhood N with the same rotation number are topologically equivalent, as are any fields in the corresponding boundaries - those with saddle connections (only we must distinguish between 'lower' and 'upper' saddle connections). The topological equivalence can be constructed in essentially the same way in all cases, as follows.

Choose $X, Y \in N$ of the same type and with the same rotation number. We first restrict to the transverse circle Σ. Call the successive inverse intersections of the stable manifolds of the sinks of X, Y with Σ, I_j, $\tilde{I}_j$ respectively. If the fields X and Y are Morse-Smale, I_j and $\tilde{I}_j$ will have two components for large enough j. The restriction of the topological equivalence to Σ is defined by taking I_j to $\tilde{I}_j$ affinely according to the ratio of their lengths (naturally we deal with the two components of I_j and $\tilde{I}_j$ separately, if necessary). Since the stable manifold of each sink is dense in Σ (cf. (1.2)) the map can be extended uniquely to the whole of Σ. Because X and Y have the same rotation number the I_j's, $\tilde{I}_j$'s intersect Σ in the same order, and so the map is indeed continuous. The map may now be extended to the whole of Σ, but care must be taken near the saddle separatrices.

For points x,y in the same orbit we let $\ell(xy)$ denote the arc length of the orbit between x and y, using the metric induced by the Euclidean metric. Then put

$$\ell(a_X S_X) = \alpha \qquad \ell(a_Y S_Y) = \hat{\alpha}$$
$$\ell(S_X t_X) = \beta \qquad \ell(S_Y t_Y) = \hat{\beta}$$
$$\ell(S_X P_X) = \gamma \qquad \ell(S_Y P_Y) = \hat{\gamma}$$
$$\ell(b_X S_X) = \delta \qquad \ell(b_Y S_Y) = \hat{\delta} \qquad \text{(see Figure 9).}$$

Let $x \in \Sigma$ be close to, and below, a_X. Then the arc of orbit $xf_X(x)$ is mapped onto an arc $yf_Y(y)$ determined by the map on Σ. Let $\ell(xf_X(x)) = r$ and $\ell(yf_Y(y)) = s$. We split $xf_X(x)$ into two parts of length $\frac{r\alpha}{\alpha+\beta}$ and $\frac{r\beta}{\alpha+\beta}$ and $yf_Y(y)$ into two parts of length $\frac{s\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$ and $\frac{s\hat{\beta}}{\hat{\alpha}+\hat{\beta}}$. The first and

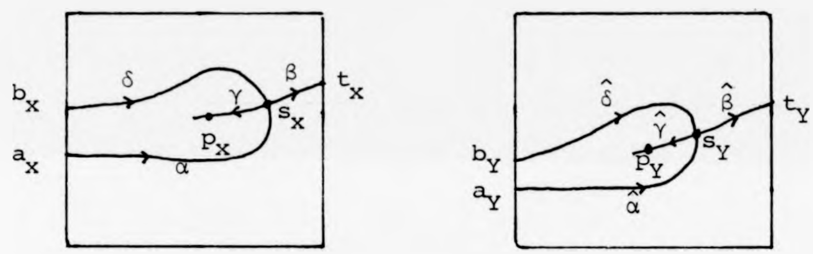


Figure 9.

second parts of $xf_X(x)$ are then mapped onto the corresponding parts of $yf_Y(y)$ according to ratio of arc length. For points close to, and above, a_X we do a similar procedure , this time splitting into parts of ratio $\frac{\alpha}{\alpha+\gamma}$ and $\frac{\gamma}{\alpha+\gamma}$ for X and of ratio $\frac{\hat{\alpha}}{\hat{\alpha}+\hat{\gamma}}$ and $\frac{\hat{\gamma}}{\hat{\alpha}+\hat{\gamma}}$ for Y. We then do a similar procedure for points close to b_X. Finally for points away from a_X and b_X we map arcs of trajectories according to their whole lengths between intersections of Σ and smooth these separate maps together by a partition of unity subordinate to a suitable cover of Σ.

Recall now the path $\phi \in C^1(I,N)$ examined in §2 and defined so that $f_{\phi(t)} = f_{\phi(0)} + t$. ϕ intersects each submanifold of Cherry fields or fields with a saddle connection at exactly one point. By the above remarks, to show that ϕ is mildly stable it suffices to show that a nearby path also meets every submanifold exactly once. This is true because ϕ intersects every submanifold transversely and furthermore when these submanifolds are close, they are C^1-close. Precisely, recall the maps $g_\alpha : N \to S^1$ defined in §2 for any $\alpha \in [0,1)$. g_α takes the submanifold with rotation number α onto 0 (for α rational, g_α is really two maps). These maps tell us how much to add to the Poincaré map of the field to get the right rotation number α. Then it follows that $g_\alpha \circ \phi = -\mathrm{id} + K_\alpha$ where $K_\alpha = \phi^{-1}(g_\alpha^{-1}(0))$. So in particular this shows ϕ is transverse to the submanifold $g_\alpha^{-1}(0)$. Furthermore, as already noted if g_α and g_β are close in C^0 sense, they are also close in the C^1 sense. Hence if $t_0 = (g_\alpha \circ \phi)^{-1}(0)$ there are neighbourhoods V_α of ϕ in $C^1(I,N)$ and U_α of t_0 in I such that if $\tilde{\phi} \in V_\alpha$ then $\tilde{\phi}$ intersects exactly once the same submanifolds as ϕ does, for $t \in U_\alpha$. We may find such a U_α for every $t \in E$, the bifurcation set of ϕ and then take a finite cover of E by U_α's, $\{U_1,\ldots,U_n\}$. Then $V = \bigcap_{i=1}^{n} V_i$ is a neighbourhood of ϕ such that $\tilde{\phi} \in V$ crosses every submanifold exactly once. Thus ϕ is mildly stable.

We now show that the topological equivalence changes continuously with t. First note that it is sufficient to do this on the restriction to Σ. So choose $\tilde{\phi} \in V$. We reparametrize $\tilde{\phi}$ by mapping the parameter linearly between corresponding bifurcation points. Let q_t be the topological equivalence between $\phi(t)$ and $\tilde{\phi}(h(t))$, where h is this reparametrizing homeomorphism. Fix $t_0 \in I$. Let $I_1, I_2, \ldots,$ and $\tilde{I}_1, \tilde{I}_2, \ldots$ be the successive inverse intersections with Σ of the stable manifolds of $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ respectively. Again, these 'intervals' may have two components if $\phi(t_0)$ is Morse-Smale. Let $\varepsilon > 0$. We consider the three cases:

(a) $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ are Cherry fields. Choose N so large that $I_1, \ldots, I_N$ and $\tilde{I}_1, \ldots, \tilde{I}_N$ both have total lengths at least $1 - \varepsilon/4$. Then choose δ so small that if $|t-t_0| < \delta$, the corresponding intervals for $\phi(t)$, $I_{1,t}, \ldots, I_{n,t}$, remain disjoint and let $M = \sup\limits_{\substack{|t-t_0|<\delta \\ 1<j<N}} \dfrac{|\tilde{I}_{t,j}|}{|I_{t,j}|}$. Then choose $\eta < \delta$ so small that the boundary points of $I_{t,1}, \ldots, I_{t,N}$ and $\tilde{I}_{t,1}, \ldots, \tilde{I}_{t,N}$ do not change by more than $\varepsilon/4M$ while $|t-t_0| < \eta$. Then $|t-t_0| < \eta \Rightarrow \|q_{t_0} - q_t\| < \varepsilon$ (see below).

(b) $\phi(t_0)$ and $\tilde{\phi}(h(t_0))$ are Morse-Smale. Choose N as in case a). Choose δ so small that if $|t-t_0| < \delta$ then $\phi(t)$ is in the same Morse-Smale class . Put $M = \sup\limits_{\substack{|t-t_0|<\delta/2 \\ 1<j<N}} \dfrac{|\tilde{I}_{t,j}|}{|I_{t,j}|}$

and $\eta < \delta/2$ so small that for $|t-t_0| < \eta$ the endpoints of $I_{t,1},\ldots,I_{t,N}$ and $\tilde{I}_{t,1},\ldots,\tilde{I}_{t,N}$ do not change by more than $\varepsilon/4M$. Then $|t-t_0| < \eta \Rightarrow \|q_{t_0}-q_t\| < \varepsilon$ (see below).

(c) $\phi(t_0)$ and $\phi(h(t_0))$ have saddle connections. Choose N and M as in case (a). $\phi(t_0)$ is on the boundary of a certain Morse-Smale class. For $t < t_0$ suppose $\phi(t)$ fails to fall into this class. Then we may choose δ so small that for $|t-t_0| < \delta, I_{t,1}, I_{t,2},\ldots I_{t,N}$ and $\tilde{I}_{t,1},\ldots,\tilde{I}_{t,N}$ remain disjoint and their endpoints move by no more than $\varepsilon/4M$. On the other hand, for $t > t_0$, $\phi(t)$ is in the Morse-Smale class. Choose γ so that for $|t-t_0| < \gamma$ the large components of $I_{t,1},\ldots,I_{t,N}$ and $\tilde{I}_{t,1},\ldots,\tilde{I}_{t,N}$ still have total length not less than $1-\varepsilon/2$, and their endpoints move by no more than $\varepsilon/4M$. Put $\eta = \min(\delta,\gamma)$. Then $|t-t_0| < \eta \Rightarrow \|q_{t_0}-q_t\| < \varepsilon$.

To see this final step in each case, consider any $x \in \Sigma$. It must satisfy one of the following:

(i) x remains outside any interval. Then $q_t(x)$ is outside any interval so is constrained to move by not more than $\varepsilon/4 + \varepsilon/4M$ in case (a) or (b), or $\varepsilon/2 + \varepsilon/4M$ in case (c) .

(ii) x remains inside a single interval. Consider Figure 10. The graph of q_t for $|t-t_0| < \eta$ connects two points in the boxes. It is not hard to see that for the worst possible x we have $|q_t(x)-q_{t_0}(x)| < 2.\varepsilon/4M + M.\varepsilon/4M < \varepsilon$.

(iii) The endpoint of an interval crosses x. Call e_t the

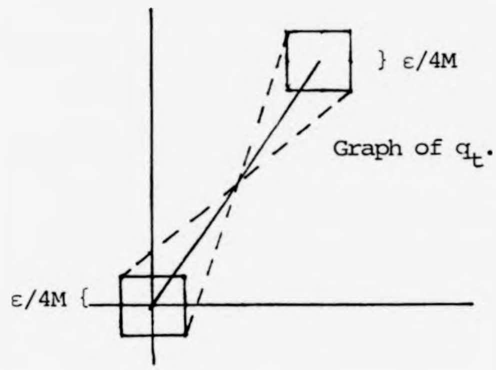


Figure 10.

endpoint directly below x and suppose it was above x for $t = t_0$. The $d(x,e_{t_0}) < \varepsilon/4M$ and $d(x,e_t) < \varepsilon/4M$. In cases (a) or (b) we have $d(q_{t_0}(x),\tilde{e}_{t_0}) < M.\varepsilon/4M + \varepsilon/4$ and $d(q_t(x),\tilde{e}_t) < \varepsilon/4$, where $\tilde{e}_t$ is the corresponding endpoint for $\tilde{\phi}(t)$. Since we also have $d(e_{t_0},e_t) < \varepsilon/4M$ it follows that $d(q_{t_0}(x),q_t(x)) < \varepsilon/2 + \varepsilon/4 + \varepsilon/4M < \varepsilon$ (see Figure 11). In case (c) the same argument holds, except we may need to replace $\varepsilon/4$ by $\varepsilon/2$ in one place.

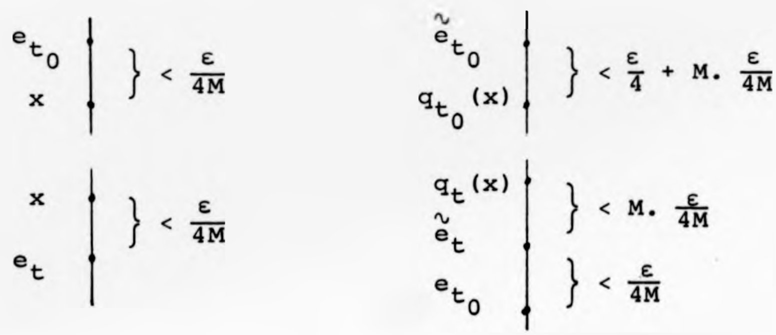


Figure 11.

Thus we have shown that q_t is continuous in t and so ϕ is a stable path as claimed.

CHAPTER 2

Symbolic dynamics and rotation intervals for endomorphisms of the circle.

"... ere the circle homeward hies
Far, far it must remove."

The concept of rotation number for homeomorphisms of the circle goes back to Poincaré and is now well known. In [16] it was generalized, for degree one endomorphisms, to a rotation set which was shown in [13] to be a closed interval. Recently it was shown in [3] how this interval is made up in terms of the rotational behaviour of each point of the circle. In this paper we use shift spaces to give alternative proofs of the results of [3] and [13] and give additional information about how the individual behaviour of points may vary around the circle.

§1. Definitions and Statements of Results

Let f be a continuous map of the circle to itself of degree one. We may choose a lift $F:\mathbb{R} \to \mathbb{R}$ of f so that $f\circ\pi = \pi\circ F$, where $\pi:\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is the usual projection. Note that F is defined uniquely up to addition of an integer.

(1.1) Definition The rotation set of f is given by

$$\rho(f) = \{\rho^+(f,z) \mid z \in S^1\} ,$$

where, with $x \in \pi^{-1}(z)$,

$$\rho^+(f,z) = \limsup_{n\to\infty} \frac{F^n(x)-x}{n} .$$

The rotation set of z under f is given by

$\rho(f,z) = \{$Limit points of the sequence $\{(F^n(x)-x)/n\}_{n=1}^{\infty}$ for $x \in \pi^{-1}(z)\}$.

In the next section we give a proof of the following remarkable result.

(1.2) Theorem 1 (Baman, Malta, Pacifico, Takens)

Let $[\alpha,\beta]$ be a subinterval of $[\inf \rho(f), \sup \rho(f)]$. Then for some $z \in S^1$, $\rho(f,z) = [\alpha,\beta]$.

The proof is first given for $[\inf \rho(f), \sup \rho(f)]$ of the form $[0,n-1]$, by modelling f with the full shift on the symbols $0,1,\ldots,n-1$, and then extended to the general case. We deduce the following corollaries:

(1.3) Corollary 1 (Ito)

The set $\rho(f)$ is a closed interval.

(1.4) Corollary 2

$$\rho(f) = \{\lim_{n\to\infty} \inf(F^n(x)-x)/n \,|\, x \in \mathbb{R}\}$$

and $$\rho(f) = \{\lim_{n\to\infty} (F^n(x)-x)/n \text{ for those } x \text{ for which the limit exists}\}.$$

(1.5) Corollary 3

If $[\alpha,\beta] \subset \rho(f)$ then there exist uncountably many points $z \in S^1$ with $\rho(f,z) = [\alpha,\beta]$.

In the third section we show how to construct a general shift space for any endomorphism, and use Birkhoff's Ergodic Theorem to prove the following result.

(1.6) Theorem 2

Suppose that f leaves invariant a probability measure that is equivalent to Lebesgue measure. Then, for almost every z in S^1 (with respect to Lebesgue measure), $\rho(f,z)$ is a single point.

We note that by Adler's Theorem ([1],[6]) many such examples exist. We give a particular example for which $\rho(f,z) = 2/3$ for almost every z, but for which, given $[\alpha,\beta] \subset [0,1]$, the set of z with $\rho(f,z) = [\alpha,\beta]$ is dense in S^1. Using results of Volkmann ([22]) on p-adic expansions we also give estimates of the Hausdorff dimension of $\{z|\rho(f,z) = [\alpha,\beta]\}$ for this example.

In the final section we show that for any continuous map of the circle of degree one, the circle is divided into intervals of two kinds. On the first kind $\rho(f,z)$ is constant and on the second kind every possible $\rho(f,z)$ is represented densely. Finally we give an idea of how $\{(F^n\text{-id})/n\}$ converges on the second kind of interval.

§2. Constructing Shift Spaces and Proof of Theorem 1.

We prove Theorem 1 in three stages. Firstly it is shown that if $\rho(f)$ contains the integers $\{0,1,\dots,n-1\}$ then each $[\alpha,\beta] \subset [0,n-1]$

is represented as some $\rho(f,z)$. This is done by showing that f 'contains' the full shift on the n symbols $0,1,\ldots,n-1$. Next this is extended to the case where $\rho(f)$ contains rationals p_1/q_1 and p_2/q_2 and $[\alpha,\beta] \subset [p_1/q_1, p_2/q_2]$. Finally the full theorem is proved by a limiting argument.

We need two known results. The following is proved in [16, p.44].

(2.1) <u>Lemma</u>

Suppose that f has no periodic point of rotation number p/q and period q. Then either

$$\rho(f) \subset \{x \in \mathbb{R} \mid x < p/q\} \quad \text{or} \quad \rho(f) \subset \{x \in \mathbb{R} \mid x > p/q\} \ .$$

We sketch a proof of the following result from [3].

(2.2) <u>Lemma</u>

For all $z \in S^1$, $\rho(f,z)$ is a closed interval.

<u>Proof</u>

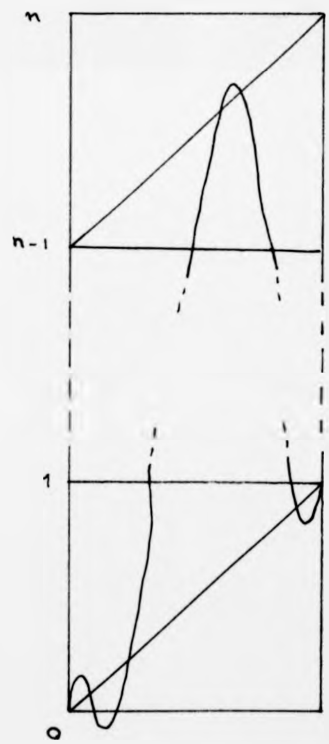
If we put $a_i = F^i(x) - F^{i-1}(x)$ then $\frac{F^n(x)-x}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i$ and the a_i are bounded. Clearly $\rho(f,z) \subset [\liminf \frac{1}{n}\Sigma a_i \ , \ \limsup \frac{1}{n}\Sigma a_i]$ and contains the endpoints. But the average must move between the endpoints by ever smaller steps so every point in the interval must be a limit point. $\square$

(2.3) Proposition

Suppose $\rho(f)$ contains the integers $0,1,\dots,n-1$. Given $[\alpha,\beta] \subset [0,n-1]$, $\alpha \le \beta$, there exists $z \in S^1$ with $\rho(f,z) = [\alpha,\beta]$.

Proof

By (2.1) we know there are fixed points for f with rotation numbers $0,1,\dots,n-1$. We may assume that a lift F of f is chosen so that $0 \in \mathbb{R}$ is a fixed point for F. This is because $\rho(R_t \circ f \circ R_t^{-1}, z) = \rho(f, z-t)$ where R_t is rigid rotation through t. Thus we may choose intervals $A_i = [a_i, b_i] \subset [0,1]$ for $0 \le i \le n-1$ satisfying:



i) $a_i < a_{i+1}$ and $b_i < b_{i+1}$ for $0 \le i \le n-2$

ii) if $x \in A_i$ then $i \le F(x) \le i+1$

iii) $F(a_i) = i$ for $0 \le i \le n-1$

and $F(b_i) = i+1$ for $0 \le i \le n-2$

while $F(b_{n-1}) = b_{n-1}$.

Let us write $\bar{A}_i = \pi(A_i)$. Then for all i,j $f(\bar{A}_i) \supset \bar{A}_j$ and furthermore $f^r(\bar{A}_{i(0)} \cap f^{-1}(\bar{A}_{i(1)}) \cap \dots \cap f^{-r}(\bar{A}_{i(r)})) = \bar{A}_{i(r)}$ for any sequence $(i(0),\dots,i(r))$; this follows since $g(X \cap g^{-1}(Y)) = g(X) \cap Y$ for any function g and sets X,Y and by using induction on r. Let us define

$$W = \{z \in S^1 \mid f^m(z) \in \bar{A}_0 \cup \bar{A}_1 \cup \ldots \cup \bar{A}_{n-1} \quad \text{for all} \quad m \geq 0\} \quad .$$

If Σ_n is the full shift on the n symbols $0,1,\ldots,n-1$ we define $P:W \to \Sigma_n$ by $(P(w))_r = i$ if $f^r(w) \in \bar{A}_i$. If $a_0 = 0$ then we will allow P to take two values at points whose orbit lands on some a_i . Apart from this possibility P is uniquely defined. We claim that P is onto Σ_n . So choose any sequence $(i(0),i(1),\ldots) \in \Sigma_n$. If we let $B_r = \bar{A}_{i(0)} \cap f^{-1}(\bar{A}_{i(1)}) \cap \ldots \cap f^{-r}(A_{i(r)})$ then $\{B_r\}_{r=0}^{\infty}$ is a nested sequence of nonempty closed subsets of S^1 . Hence by the finite intersection property there is a point z in every B_r and $P(z) = (i(0),i(1),\ldots)$. So the claim is true.

Now for any point z in W , if $P(z) = (i(0),i(1),\ldots)$ and $x \in \pi^{-1}(z)$ then $|(F^n(x)-x) - (i(0)+\ldots+i(n-1))| < 1$. This is because the sequence counts integers passed by the orbit of x but there could be a discrepancy of up to 1.

Thus we have

$$\rho(f,z) = \{\text{Limit points of } \{\tfrac{1}{r}\sum_{j=0}^{r-1} i(j)\}_{r=1}^{\infty}\} \quad .$$

Now given any $[\alpha,\beta] \subset [0,n-1]$ we may choose a sequence $\{i(r)\} \in \Sigma_n$ with the limit points of $\frac{1}{r}\Sigma i(j)$ equal to $[\alpha,\beta]$. Since P is onto Σ_n there exists $z \in S^1$ with $P(z) = \{i(r)\}$ and so $\rho(f,z) = [\alpha,\beta]$.

□

(2.4) Corollary

Suppose $0,\dots,n-1$ are in $\rho(f)$. Let $\varepsilon > 0$ and γ be either 0 or $n-1$. Then there exists an integer n_0 and a sequence of nested closed subsets $\{I_i\}$ such that if $z \in I_N$ for any integer $N > n_0$ then $\left|\frac{F^j(x)-x}{j} - \gamma\right| < \varepsilon$ for all j with $n_0 \le j \le N$, where $x \in \pi^{-1}(z)$. Furthermore, for all $j \ge 1$ $f^{j+2}(I_j) = S^1$.

Proof

Choose $n_0 > 1/\varepsilon$ and let $I_j = \bar{A}_\gamma \cap f^{-1}(\bar{A}_\gamma) \cap \dots \cap f^{-j}(\bar{A}_\gamma)$.
Then if $\pi(x) = z \in I_j$

$$\left|\frac{F^n(x)-x}{n} - \gamma\right| = \frac{1}{n}\left|(F^n(x)-x) - n\gamma\right| < \frac{1}{n} < \varepsilon$$

if $n > n_0$. Also as above we get $f^{j+2}(I_j) = f^2(\bar{A}_\gamma) = S^1$. □

(2.5) Lemma

If $\rho(f,z) = [\alpha,\beta]$ then $\rho(f^m,z) = [m\alpha, m\beta]$ for $m \ge 1$.

Proof

It is clear that $\rho(f^m,z) \subset [m\alpha, m\beta]$. Since, by (2.2), $\rho(f^m,z)$ is a closed interval it is sufficient to show that $m\alpha$ and $m\beta$ are in $\rho(f^m,z)$. We show that $m\alpha \in \rho(f^m,z)$. Let $M = \sup\limits_{\substack{x \in \mathbb{R} \\ 1 \le j \le m-1}} |F^j(x)-x|$.

Since $\alpha \in \rho(f,z)$, given $\varepsilon > 0$ there are infinitely many integers of the form $mr + j$, with $0 \le j \le m-1$, such that if $x \in \pi^{-1}(z)$ then

$$\left| \frac{F^{mr+j}(x)-x}{mr+j} - \alpha \right| < \varepsilon \quad \text{and so} \quad \left| \frac{F^{mr+j}(x)-x}{r+j/m} - m\alpha \right| < m\varepsilon \quad .$$

But

$$\left| \frac{F^{mr+j}(x)-x}{r+j/m} - \frac{F^{mr}(x)-x}{r} \right| = \left| \frac{F^{j}(F^{mr}(x)) - F^{mr}(x)}{r+j/m} + \right.$$

$$\left. \frac{F^{mr}(x)-x}{r+j/m} - \frac{F^{mr}(x)-x}{r} \right| \le \frac{M}{r+j/m} + \left| \frac{j(F^{mr}(x)-x)}{mr(r+j/m)} \right|$$

$$< \varepsilon \text{ for large enough } r \text{ .}$$

Hence, for infinitely many r , $\left| \frac{F^{mr}(x)-x}{r} - m\alpha \right| < (m+1)\varepsilon$.

Since ε was arbitrary, $m\alpha \in \rho(f^m,z)$ as required. □

Notice that (2.3) and (2.5) together prove theorem 1 in the case $[\alpha,\beta] \subset [p_1/q_1, p_2/q_2]$ if we know p_1/q_1 and p_2/q_2 are in $\rho(f)$. However we will go on to the general case. We first need a generalisation of (2.4).

(2.6) Lemma

Suppose $\{p_1/q_1, p_2/q_2\} \subset \rho(f)$. Let $\varepsilon > 0$ and γ be either

p_1/q_1 or p_2/q_2. Then there exists an integer n_0 and a sequence of nested closed sets $\{I_i\}$ such that if $z \in I_N$ for any integer $N > n_0$

then $$\left| \frac{F^{jq_1q_2}(x) - x}{jq_1q_2} - \gamma \right| < \varepsilon \quad \text{for all } j \text{ with } n_0 \le j \le N ,$$

where $x \in \pi^{-1}(z)$. Furthermore, for all $j \ge 1$, $f^{q_1q_2(j+2)}(I_j) = S^1$.

Proof

By (2.5), $\{p_1q_2, p_2q_1\} \subset \rho(f^{q_1q_2})$. Since $\rho(f)$ is only defined up to translation by integers we may apply (2.4) with F replaced by $F^{q_1q_2}$. Thus for example there is a sequence $\{I_i\}$ such that if $z \in I_N$ and $x \in \pi^{-1}(z)$ then

$$\left| \frac{F^{jq_1q_2}(x)-x}{jq_1q_2} - \frac{p_2}{q_2} \right| = \frac{1}{q_1q_2} \left| \frac{F^{jq_1q_2}(x)-x}{j} - p_2q_1 \right| < \frac{\varepsilon}{q_1q_2}$$

for $n_0 \le j \le N$ and $f^{q_1q_2(j+2)}(I_j) = S^1$. □

(2.7) Proof of Theorem 1

Let $[\alpha,\beta] \subset [\inf \rho(f), \sup \rho(f)]$. Choose a sequence of rationals $\{p_i/q_i\}$ so that $p_{2i}/q_{2i} \to \alpha$ and $p_{2i+1}/q_{2i+1} \to \beta$, and a decreasing sequence of positive numbers $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$. We choose inductively a sequence of nested closed sets $\{J_i\}$. At the i'th stage we use (2.6) with $p_1/q_1 = p_i/q_i$ and $p_2/q_2 = p_{i+1}/q_{i+1}$ (or vice versa), $\varepsilon = \varepsilon_i/2$ and $\gamma = p_{i+1}/q_{i+1}$ and put

$$J_{i+1} = f^{-(M_i+2q_iq_{i+1})}(I^{i+1}_{N(i+1)}) \cap J_i$$

where $M_i = q_1q_2(N(1)+2) + q_2q_3(N(2)+2) +\ldots+ q_iq_{i+1}N(i)$.
Then $f^{M_{i+1}+2q_{i+1}q_{i+2}}(J_{i+1}) = S^1$ and so we may go on to the next stage.
Now at the i'th stage we are free to choose $N(i)$ as large as we like so we ensure

i) $$\left| \frac{F^{M_i}(x)-x}{M_i} - \frac{p_{i+1}}{q_{i+1}} \right| < \varepsilon_i \quad \text{if } \pi(x) \in J_i$$

ii) $$\frac{F^{M_i+j}(x)-x}{M_i+j} \in [\alpha-\varepsilon_i, \beta+\varepsilon_i] \quad \text{for all } j \geq 1 .$$

By the finite intersection property there is some z in every J_i .
Since i) holds $\rho(f,z) \supset [\alpha,\beta]$ and by ii) $\rho(f,z) \subset [\alpha-\varepsilon_i,\beta+\varepsilon_i]$ so $\rho(f,z) = [\alpha,\beta]$. □

(2.8) <u>Proof of Corollaries</u>

Both (1.3) and (1.4) follow because every γ in $\rho(f)$ exists as some $\rho(f,z) = \rho^+(f,z)$. To prove (1.5) suppose to the contrary that only countably many points have a certain rotation interval $I \subset \rho(f)$. Then we may label them $z^1, z^2, \ldots$ We show that there is a point z not in this list with $\rho(f,z) = I$. We do this by choosing a nested sequence as above but ensuring during the i'th stage that $z \neq z_i$. To see how this can be done recall $f^{q_i q_{i+1}}$ contains the full shift on a certain number of symbols. Thus we may let a certain power of z fall into a different interval in the representation of this shift without disturbing i) and ii). Since a point can have at most 2 sequences in this subshift we may ensure $z \neq z_i$. □

§3. Distribution of Individual Rotation sets

(3.1) Constructing General Shift Spaces

In order to prove Theorem 2 we construct a shift space corresponding to any given endomorphism. Let F be a lift of f chosen so that $F(x) \geq 0$ for all $x \in [0,1]$ and $F(x) < 1$ for some $x \in [0,1]$. Let n be the smallest integer n satisfying $F(x) < n$ for all $x \in [0,1]$. Then for $0 \leq j \leq n-1$ we define

$$A_j = \pi\{x \in [0,1] \mid j \leq F(x) \leq j+1\} .$$

Then each A_j is closed and non empty. Define

$$\Sigma = \{(i(0),i(1),\ldots) \mid A_{i(0)} \cap f^{-1}(A_{i(1)}) \cap \ldots \neq \phi\} .$$

Thus Σ is a subset of Σ_n . We shall say that a point $z \in S^1$ is associated with a sequence $(i(0),i(1),\ldots) \in \Sigma$ if $f^r(z) \in A_{z(r)}$ for all $r \geq 0$. Then every point in S^1 is associated with a unique sequence in Σ except for those whose orbits contain $0 \in S^1$. Clearly if z is associated with $(i(0),i(1),\ldots)$ then $f(z)$ is associated with $(i(1),i(2),\ldots)$. Now if z is associated with a unique sequence $(i(0),i(1),\ldots)$ then if $x \in \pi^{-1}(z)$,

$$|(F^r(x) - x) - (i(0) + \ldots + i(r))| < 1$$

and $$\rho(f,z) = \{\text{Limit points of } \{\frac{1}{r} \sum_{j=0}^{r} i(j)\}_{r=1}^{\infty}\} .$$

(3.2) Proof of Theorem 2

Let μ be a probability measure on S^1 which is invariant under f and equivalent to Lebesgue measure λ . Construct a subshift as above and let $E = \{z \in S^1$ associated with more than one sequence$\}$. Since $E = \bigcup_{n=1}^{\infty} f^{-n}(0)$, and μ is invariant, it follows that $\mu(E) = 0$.
Thus μ is a probability measure on $Y = S^1 - E$. Define an integrable function on Y by

$$\chi(z) = i(0) \quad \text{if } z \text{ is associated with } (i(0),i(1),\ldots) .$$

By Birkhoff's Ergodic Theorem (see for example [23]) we have

$$\frac{1}{k}\sum_{i=0}^{k-1} \chi(f^i(z)) \to g(z) \qquad \text{a.e. } (\mu)$$

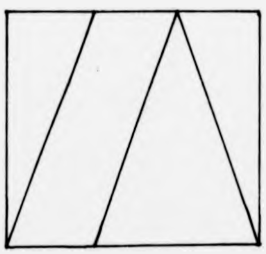
for some integrable function g . But $\frac{1}{k}\sum_{i=0}^{k-1} \chi(f^i(z)) = \frac{1}{k}\sum_{j=0}^{k-1} i(j)$.

Hence $\rho(f,z) = g(z)$ for almost every point with respect to λ . □

We note that Adler's Theorem ([6] Theorem 1.2, [1]) guarantees many examples satisfying the hypothesis of Theorem 2. Adler's Theorem states that if f is piecewise C^2 , Markov, expanding and $\sup_{x\in S^1} |f''(x)|/|f'(x)|^2 < \infty$ then f admits a unique probability measure invariant and equivalent to Lebesgue measure. In this case the measure is also ergodic which means that the function g is constant.

(3.3) <u>Example</u>

Consider the endomorphism f of the circle defined by its lift F satisfying



$$F(x) = \begin{cases} 3x & 0 \le x \le 2/3 \\ 3-3x & 2/3 \le x \le 1 . \end{cases}$$

We take the subshift defined by this lift as in (3.1). Note that f leaves Lebesgue measure λ invariant, and is ergodic with respect to λ (for example by comparison with $x \to 3x \bmod 1$). So in this case Birkhoff's Ergodic Theorem tells us that $\rho(f,z) = \int \chi\, d\lambda = 2/3$ a.e. (λ) . The subshift for f contains the full shift on $\{0,1\}$. Since any sequence containing the symbol 2 ends in infinitely many 0's we deduce that $\rho(f) = [0,1]$. Now since $\rho(f,z)$ is independent of the first finitely many coordinates in a sequence associated with z , it follows that for any $[\alpha,\beta] \subset [0,1]$ the set $\{z|\rho(f,z) = [\alpha,\beta]\}$ is dense in the circle. We show below that this behaviour is typical.

We have already noted the similarity between this example and the map $x \to 3x \bmod 1$ associated with the expansion to base 3 of the real numbers in $[0,1]$. We may exploit this by constructing a new shift space with rectangles $A_0 = [0,1/3]$, $A_1 = [1/3,2/3]$, $A_2 = [2/3,1]$. Then results about the Hausdorff dimension of sets of sequences representing real numbers in $[0,1]$, in their base 3 expansion, carry over to sets of points in S^1 represented by sequences in this new shift. This is because we need only consider covers by m-cylinders $[r/3^m, r+1/3^m]$ when calculating Hausdorff dimensions (see Billingsley [5, p.140]), and apart from the countable number of points with more than one sequence, the points corresponding to two sequences are in the same m-cylinder in the one system if and only if they are in the same m-cylinder in the other. For a point $x \in [0,1]$ let $A_i(x,n)$ = the number of i's in the first n coordinates of the ternary expansion of x , $i = 0,1,2$. Define

$\rho_i(x) = \{\text{Limit points of}\{\frac{A_i(x,n)}{n}\}_{n=1}^{\infty}\}$. So $\rho_i(x)$ is a closed subinterval of $[0,1]$. Choose any set V of the form $\rho_0(x) \times \rho_1(x) \times \rho_2(x)$. Then Volkmann [22] showed that, if $\dim_H$ denotes Hausdorff dimension,

$$(3.3) \qquad \dim_H\{y|\rho_0(y) \times \rho_1(y) \times \rho_2(y) = V\} = \inf_{\underline{y}\in V} d(\underline{y})$$

where d is given by $d(\underline{y}) = \frac{\sum_{i=0}^{2} y_i \log y_i}{\log(1/3)}$ for $\underline{y} = (y_0,y_1,y_2) \in V$.

If the $\rho_i(x)$'s are single numbers with $\rho_0(x) + \rho_1(x) + \rho_2(x) = 1$, (3.3) gives the Hausdorff dimension of the set of generic points for the Bernoulli measure on the full 3-shift determined by $(\rho_0(x),\rho_1(x),\rho_2(x))$. Returning to our example note that

$\rho(f,z) = \{\text{Limit points }\{\frac{A_1(z,n) + A_2(z,n)}{n}\}_{n=1}^{\infty}\}$. To find points $z \in S^1$ with $\rho(f,z) = [\alpha,\beta]$ we must consider sequences with $\rho_0(z) = [1-\beta,1-\alpha]$ and $\rho_1(z),\rho_2(z)$ may take various values. From (3.3) we deduce that $\{z \in S^1|\rho(f,z) = [\alpha,\beta]\}$ has positive Hausdorff dimension as long as $\alpha \neq 0$ and $\beta \neq 1$. In general (3.3) only gives a lower bound. For example, let $U = \{z|\rho(f,z) = \{\frac{1}{2}\}\}$. Any set of the form $\{y | p_0(y), p_1(y), p_2(y) = (\frac{1}{2}, p, \frac{1}{2}-p)\}$ is inside U, for any $p \in [0,\frac{1}{2}]$. Since $d(y)$ is maximum when $p = \frac{1}{4}$ we have $\dim_H U \geq (3 \log 2)/(2 \log 3)$.

§4. The General Case

We noted that for the above example the set of points with a given rotation interval is dense. The following shows that this is always the case away from intervals of z where $\rho(f,z)$ is constant.

(4.1) Proposition

Suppose $\rho(f,z_1) \neq \rho(f,z_2)$. Then given $[\alpha,\beta] \subset \rho(f)$ there exists y with $z_1 < y < z_2$ and $\rho(f,y) = [\alpha,\beta]$.

Proof

Let $\gamma \in \rho(f,z_1)$ with $\gamma \notin \rho(f,z_2)$. Let $\inf_{\delta \in \rho(f,z_2)} |\gamma - \delta| = \varepsilon$.

Then for infinitely many r we have $\left| \frac{F^r(w)-w}{r} - \gamma \right| < \frac{\varepsilon}{4}$ and $\left| \frac{F^r(x)-x}{r} - \gamma \right| > 3\varepsilon/4$, if $w \in \pi^{-1}(z_1)$ and $x \in \pi^{-1}(z_2)$. Thus $|F^r(x) - F^r(w)| > r\varepsilon/2 - |x-w| > 1$ for large enough r. That is $f^r([z_1,z_2]) = S^1$. But since $\rho(f,z) = \rho(f,f^n(z))$ for any $n \geq 1$, the result follows by Theorem 1.

It is easy to construct examples of maps for which $\rho(f,z)$ is constant for an interval of z ; for instance take any map which is itself constant on an interval. Away from such intervals the average behaviour of points converges in the following way.

(4.2) Proposition

Let $I \subset S^1$ be an interval such that $\rho(f,z)$ is not constant on any nontrivial subinterval of I . Then

$$d_H \left(\text{Graph } \left(\frac{F^n - id}{n}\right)\Big|_I \; , \; I \times \rho(f)\right) \to 0 \quad \text{as} \quad n \to \infty$$

where d_H denotes the Hausdorff metric on closed subsets of $\mathbb{R}^2$.

Proof

We must show that for any $\varepsilon > 0$, Graph$(F^n\text{-id})/n$ is ε-dense in $I \times \rho(f)$ for large enough n . By (4.1) we may choose a finite number of points $z_1,\ldots,z_s \in I$ with $\rho(f,z_i)$ alternately equal to the single points $\inf \rho(f)$ and $\sup \rho(f)$, and wemay choose the z_i's as close as we please. Then for close enough z_i's and large enough n , Graph $(F^n\text{-id})/n$ is ε-dense.

It remains to show that no points outside $I \times \rho(f)$ is an accumulation point of $\{\text{Graph } (F^n\text{-id})/n\}_{n=1}^{\infty}$. Suppose, for a contradiction, that there is such a point z . Then there exists $\varepsilon > 0$ so that for infinitely many n and z_n, $(F^n(z_n)-z_n)/n > \sup \rho(f) + \varepsilon$. Now there is a small interval I_n around each z_n so that if $z \in I_n$ then $(F^n(z)-z)/n > \sup \rho(f) + \varepsilon/2$ and furthermore there exists $r(n)$ so that $f^{r(n)}(I_n) = S^1$. Hence we may inductively choose a nested sequence of closed subsets $\{J_i\}$ and a subsequence $I_{n(i)}$ so that if $z \in J_i$ then $f^{M_i}(z) \in I_{n(i)}$, where $M_i = n(1) + r(1) + \ldots + n(i-1) + r(i-1)$. At the i'th stage we choose $n(i)$ so large that if $z \in J_i$ then

$$\frac{F^{M_i+n(i)}(z) - z}{M_i + n(i)} > \sup \rho(f) + \frac{\varepsilon}{4} .$$

If $z \in \bigcap_{i=1}^{\infty} J_i$ then $\sup \rho(f) + \varepsilon/4 \in \rho(f,z)$. This contradiction completes the proof. $\square$

CHAPTER 3

Paths through endomorphisms of the circle.

"The happy highways where I went . . ."

§1. Statement of Result

In [16] the following conjecture was made:

(1.1) Conjecture (Newhouse, Palis, Takens)

Let $t \to f_t$, $t \in [0,1]$, be a path of class C^1 in the space of C^∞ endomorphisms of the circle. Define $E = \{t \mid$ the endpoints of the rotation interval of f_t are irrational$\}$. As long as the rotation interval changes at all along the path then $m(E) > 0$, where m denotes Lebesgue measure.

If true, this would generalise the theorem of Herman for diffeomorphisms (1.2.2). However recent numerical work of Lanford [14] suggests that (1.1) may be false. Using a computer he estimates that for the map

$$x \to x + (1/2\pi) \sin 2\pi x + t$$

we have $m(E) = 0$ where E is defined as in (1.1). Of course this map is a homeomorphism and so each rotation interval is a single point.

In this chapter we use some of the results in Chapters 1 and 2 to prove the following which shows that (1.1) does not hold for certain piecewise C^1 maps.

(1.2) Theorem

Let f be a continuous degree one map of the circle and suppose the circle is partitioned into two intervals I_1 and I_2 such that:

1) f is C^1 on I_1 and $f'(x) > \lambda > 1$ for all $x \in I_1$.

2) f is monotonic decreasing on I_2 .

Let $f_t = R_t o f$ where R_t is rigid rotation through angle $2\pi t$.

Define $E = \{t \in [0,1]$ the endpoints of the rotation interval of f_t are irrational$\}$.

Then $m(E) = 0$ and furthermore E has zero Hausdorff dimension.

This theorem is of course a generalisation of (1.1.3).

§2. Proof of (1.2)

The idea of the proof is to construct monotonic maps g_t and h_t with $\rho(g_t) = \inf \rho(f_t)$ and $\rho(h_t) = \sup\ (f_t)$ and then apply (1.1.3) to these maps. Let F be a lift of f , and $\pi: \mathbb{R} \to S^1$ the usual projection so that $\pi o F = f o \pi$.

(2.1) Definition

Let $[a,b]$ be an interval in $\mathbb{R}$ with $b < a+1$. Call $\pi([a,b]) \subset S^1$ a peak (or valley) for f if $F(a) = F(b)$ and $F(x) > F(a)$ (or $F(x) < F(a)$) for all $x \in (a,b)$. Call $\pi([a,b])$

a <u>plateau</u> if $F(x) = F(a)$ for all $x \in [a,b]$.

<u>(2.2) Proposition</u>

Suppose f has a peak or a valley $[z_1,z_2] \subset S^1$. Let $\hat{f}$ be the unique endomorphism of the circle which has a plateau at $[z_1,z_2]$ and is equal to f elsewhere. Then $\rho(\hat{f}) \subset \rho(f)$.

<u>Proof</u>

Choose $z \in S^1$. Recall the rotation set of z under f , $\rho(f,z)$ (see (2.1.1)). Then it is sufficient to show that $\rho(\hat{f},z) \subset \rho(f)$. If the orbit of z under $\hat{f}$ never enters $[z_1,z_2]$ then $f^n(z) = \hat{f}^n(z)$ for all n 0 and $\rho(\hat{f},z) = \rho(f,z) \subset \rho(f)$. Similarly if z enters $[z_1,z_2]$ just once, say $\hat{f}^r(z) \in [z_1,z_2]$ then $\hat{f}^{n+r}(z) = f^n(\hat{f}^r z))$ for all $n \geq 1$ and so $\rho(\hat{f},z) = \rho(\hat{f},\hat{f}^{r+1}(z)) = \rho(f,\hat{f}^{r+1}(z)) \subset \rho(f)$. The remaining case is when the orbit of z under $\hat{f}$ enters $[z_1,z_2]$ more than on a . Then the orbit of z eventually coincides with that of the point $\hat{f}([z_1,z_2])$ which is periodic since its orbit lands in $[z_1,z_2]$ again. Suppose this point has period q . By considering $\hat{f}^q$ it is clear that there is a point $y \in S^1$ with the same period and rotation number as $\hat{f}([z_1,z_2])$ but whose orbit is disjoint from the interior of $[z_1,z_2]$. Then $\rho(\hat{f},z) = \rho(\hat{f},y) = \rho(f,y) \subset \rho(f)$. □

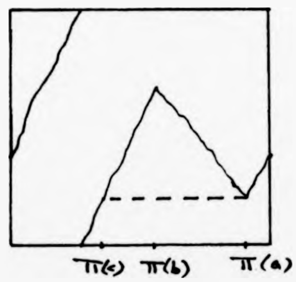
<u>(2.3) Remark</u>

Recently we have discovered that the idea of using 'surgery' to replace endomorphisms by monotonic maps has been used before; see [7] and references therein. Note that (2.2) is true for any continuous degree one map f .

(2.4) Proof of (1.2)

Let $[a,b]$ and $[b,a+1]$ be intervals in $\mathbb{R}$ such that $\pi([a,b]) = I_1$ and $\pi([b,a+1]) = I_2$. Since F is increasing on I_1, there is a unique point $c \in [a,b]$ with $F(c) = F(a+1)$. Then $\pi([c,a+1])$ is a peak for f . Let g be the unique map with a plateau at $\pi([c,a+1])$ and equal to f elsewhere. See figure 1.

Figure 1.



Now by (2.2) $\rho(g) \subset \rho(f)$ and since g is monotonic and $g \le f$ we must have $\rho(f) = \inf \rho(f)$. Clearly if $g_t = R_t \circ g$ then $\rho(g_t) = \inf \rho(f_t)$ for all t . Now g is a monotonic map with a constant interval $\pi([b,a+1])$ and it satisfies the hypotheses of (1.1.3). Hence it follows that if $E = \{t \in [0,1] \mid \inf \rho(f_t) \text{ is irrational}\}$ then $m(E) = 0$ and E has zero Hausdorff dimension. By a similar construction we may produce h with $\rho(h_t) = \sup \rho(f_t)$ and (1.2) follows. □

"Think no more, lad"

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